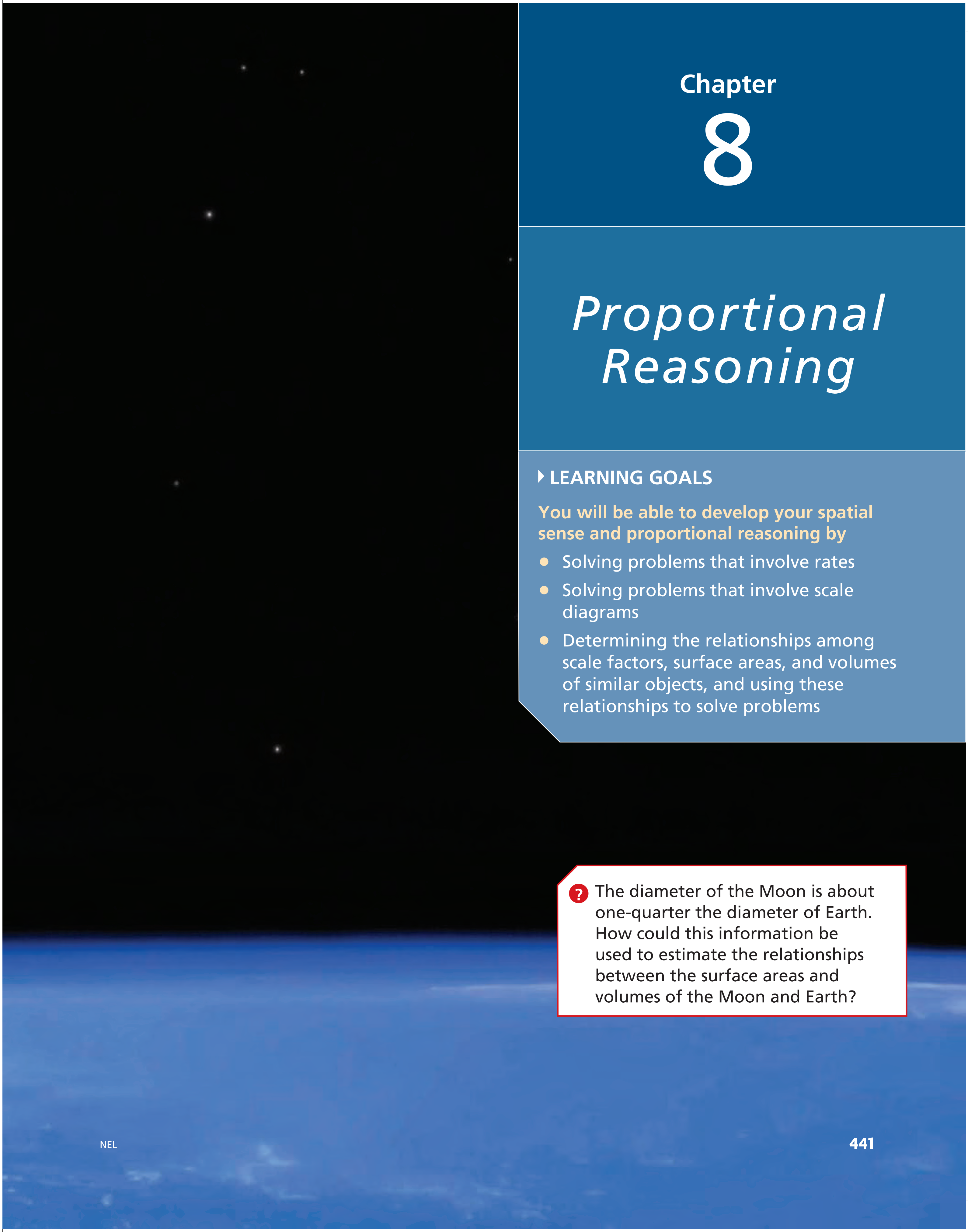


Chapter 8

Proportional Reasoning

LEARNING GOALS

You will be able to develop your spatial sense and proportional reasoning by

- Solving problems that involve rates
- Solving problems that involve scale diagrams
- Determining the relationships among scale factors, surface areas, and volumes of similar objects, and using these relationships to solve problems

? The diameter of the Moon is about one-quarter the diameter of Earth. How could this information be used to estimate the relationships between the surface areas and volumes of the Moon and Earth?

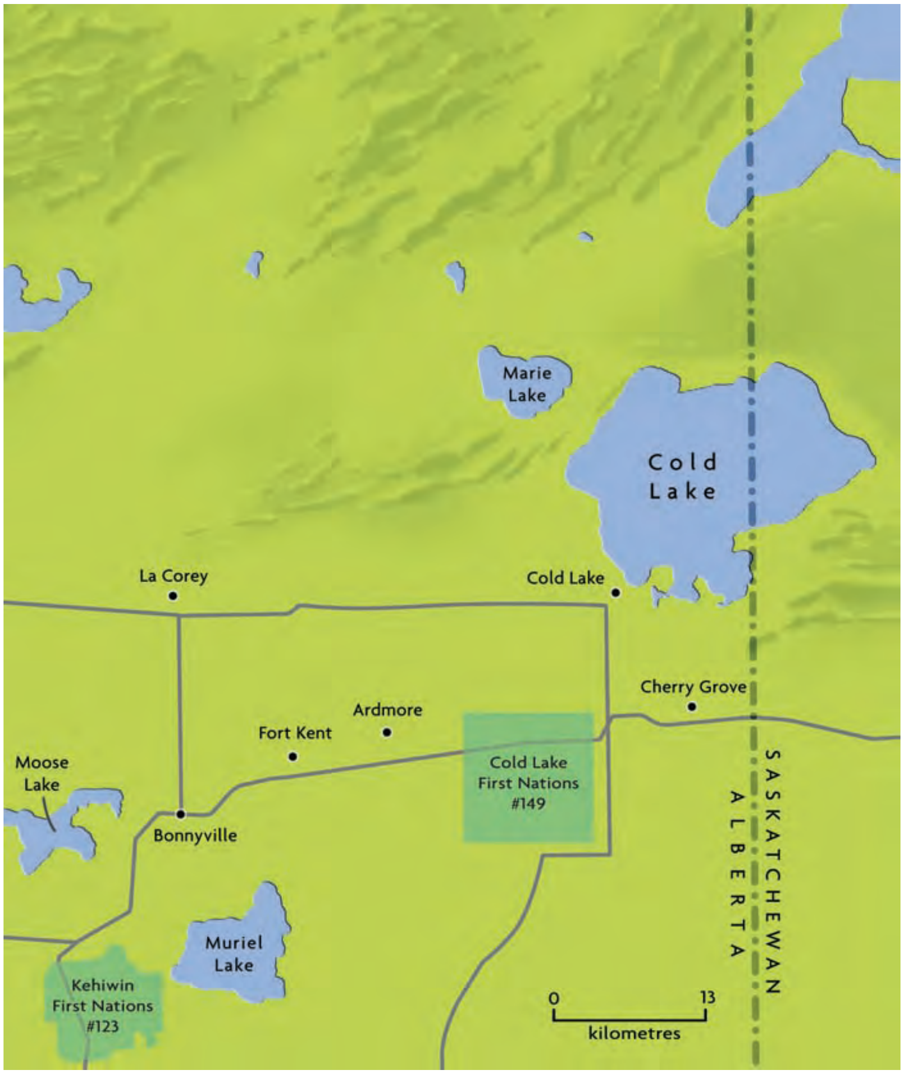
8 Getting Started

YOU WILL NEED

- calculator
- measuring tape or ruler

Interpreting the Cold Lake Region

Cold Lake is a city in Alberta's Lakelands, near its border with Saskatchewan. The following map shows information you could get from an Internet search. If you did not know the scale of the map, however, you would have difficulty judging distances and the sizes of specific locations.



The scale of this map is 2 cm equals 13 km.

? What more can you learn about the Cold Lake region from this map?

Among the prominent citizens of Cold Lake is Alex Janvier, a Dene artist born on Le Goff Reserve, Cold Lake First Nations. *Morning Star*, one of his most celebrated works, is displayed in the Canadian Museum of Civilization in Gatineau, Québec.

A. French exchange students are taking a field trip to visit some points of interest in the Cold Lake region. They are starting their trip in Bonnyville. While waiting for the bus Marie and some of her classmates go into the District Historical Museum to see the Poitras Collection of French folk art. While she is in the museum, the bus arrives. The bus is parked 300 m from the museum, and it is going to leave in 5 min. Can Marie catch the bus if she walks? Explain.

B. The first stop on the field trip is a horse farm and equestrian centre in La Corey. Estimate the distance that the bus must travel by road.

C. From La Corey, the students are going to the Alex Janvier Art Gallery in Cold Lake. Estimate the distance that the bus must travel by road.

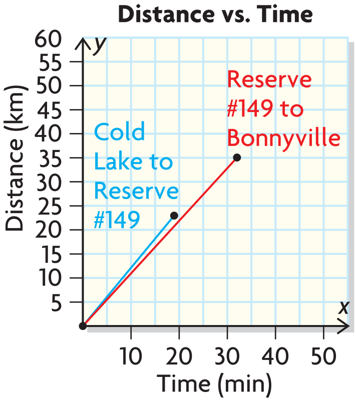
D. In Alex Janvier's *Morning Star*, there are four distinct areas of colour in the outside ring. Each area of colour represents a period in First Nations history. The total area of the painting is about 418 m^2.

- **i)** What is the area of one quadrant of the painting?
- **ii)** Describe an area in your school or community that is about the same size as one quadrant of the painting.

E. On the way back to Bonnyville, the students will visit Cold Lake First Nations Reserve #149. Estimate the area of the reserve.

F. The graph to the right shows the distance travelled over time on the last two legs of the students' field trip.

- **i)** On which leg was the bus travelling faster? Explain how you know.
- **ii)** Estimate the difference in the speeds of the bus on these two legs of the trip.



***WHAT** DO You Think?*

Decide whether you agree or disagree with each statement. Explain your decision.

1. Joe walked 5 km in 2 h, and Steff walked 3 km in 1.25 h. The only way to compare these rates is to express each rate numerically, as a unit rate.
2. Natalie needs to buy ground beef so that she can make lasagna for a party. There are two supermarkets and a butcher in her town. Price is the only factor that she needs to consider when deciding where to buy the ground beef.
3. When a 2-D shape or a 3-D object is enlarged or reduced, all of its measurements are affected by the scale factor in the same way.

8.1

Comparing and Interpreting Rates

YOU WILL NEED

- calculator
- graph paper
- ruler

EXPLORE...

- World-class sprinters can run 100 m in about 9.8 s. If they could run at this rate for a longer period of time, estimate how far they could run in a minute, in an hour, and in a day.

Communication *Tip*

The time 01:20:34.7, or 80:34.7, can also be written as 1 h 20 min 34.7 s or as 80 min 34.7 s.

GOAL

Represent, interpret, and compare rates.

INVESTIGATE the Math

A triathlon consists of three different races: a 1.5 km swim, a 40 km bike ride, and a 10 km run. The time it takes to "transition" from one race to the next is given as "Trans" in the official records. The setup for the transition area is different in different triathlons.

Simon Whitfield of Victoria, British Columbia, participated in the triathlon in both the 2000 and 2008 Olympic Games. His results for both Olympics are shown below.

2000 Sydney Olympic Games Triathlon: Men

			Swimming		Cycling		Running	
Rank	**Athlete**	**Country**	**Time**	**Trans**	**Time**	**Trans**	**Time**	**Total Time**
1	Whitfield	Canada	17:56	0:21	58:54	0:17	30:52	1:48:24.02

2008 Beijing Olympic Games Triathlon: Men

			Swimming		Cycling		Running	
Rank	**Athlete**	**Country**	**Time**	**Trans**	**Time**	**Trans**	**Time**	**Total Time**
2	Whitfield	Canada	18:18	0:27	58:56	0:29	30:48	1:48:58.47

How can you compare Simon's speeds in his two Olympic medal-winning triathlons?

A. Compare Simon's times for each race segment in the two triathlons, ignoring the transition times. Which race segment had the greatest difference in Simon's times?

B. Speed, the ratio of distance to time, is an example of a **rate**. Create a distance versus time graph to compare Simon's swimming speeds in these triathlons. Express time in hours and distance in kilometres.

rate

A comparison of two amounts that are measured in different units; for example, keying 240 words/8 min

C. Carmen claims that Simon swam faster in the 2000 Sydney Olympic Games Triathlon. Explain how the data in the table and in the graph you created in part B support her claim.

D. Determine the slope of each line segment on your distance versus time graph. What do these slopes represent?

Communication *Tip*

The word "per" means "to each" or "for each." It is written in units using a slash (/).

E. Do the slopes you determined in part D support Carmen's claim? Explain.

Reflecting

F. Can the slopes of line segments on a graph be used to compare rates? Explain.

G. Are the slopes of the line segments on your distance versus time graph **unit rates**? Explain.

H. Can the swimming speeds in each triathlon be calculated directly from the information given? Explain.

unit rate

A rate in which the numerical value of the second term is 1; for example, keying 240 words/8 min expressed as a unit rate is 30 words/min.

APPLY the Math

EXAMPLE 1 Comparing two rates expressed in different units

Natasha can buy a 12 kg turkey from her local butcher for \$42.89. The local supermarket has turkeys advertised in its weekly flyer for \$1.49/lb. There are about 2.2 lb in 1 kg. Which store has the lower price?

Natasha's Solution: Comparing using estimation

Butcher:

A 12 kg turkey from the butcher costs about \$43.

Supermarket:

The price of a turkey from the supermarket, T, is

$$T = \left(\frac{\$1.50}{1\ \cancel{\text{lb}}}\right)\left(\frac{2\ \cancel{\text{lb}}}{1\ \cancel{\text{kg}}}\right)(12\ \cancel{\text{kg}})$$

$$T = \$36$$

A 12 kg turkey from the supermarket costs about \$36.

To make a comparison, I needed to estimate the prices of two turkeys that are the same size, measured in the same units.

I multiplied the price per pound by the conversion rate for pounds to kilograms, which is approximately 2 lb/1 kg, and then I multiplied by 12 kg, the mass of the turkey. This gave me an estimate of the price of a similar turkey from the supermarket.

A turkey from the butcher costs $\frac{\$43}{12\text{ kg}}$, and a turkey from the supermarket costs $\frac{\$36}{12\text{ kg}}$. So, the price of a 12 kg turkey is about \$7 less at the supermarket.

Dimitri's Solution: Comparing using unit rates

Butcher:

The price per kilogram for a turkey from the butcher, B, is

$$B = \frac{\$42.89}{12\text{ kg}}$$

$$B = \frac{\$3.574...}{1\text{ kg}}$$

To make a comparison, I expressed each price as a unit rate, using the same units.

I divided the price of a 12 kg turkey from the butcher by the mass. This gave me the price per kilogram for a turkey from the butcher.

The price per kilogram for a turkey from the butcher is \$3.57/kg.

Supermarket:
The price per kilogram for a turkey from the supermarket, S, is

$$S = \left(\frac{\$1.49}{1\ \cancel{\text{lb}}}\right)\left(\frac{2.2\ \cancel{\text{lb}}}{1\ \text{kg}}\right)$$

$$S = \frac{\$3.278}{1\ \text{kg}}$$

I multiplied the price/lb by the conversion rate for pounds to kilograms, which is approximately 2.2 lb/1 kg. This gave me the price per kilogram for a turkey from the supermarket.

The price per kilogram for a turkey from the supermarket is \$3.28/kg.

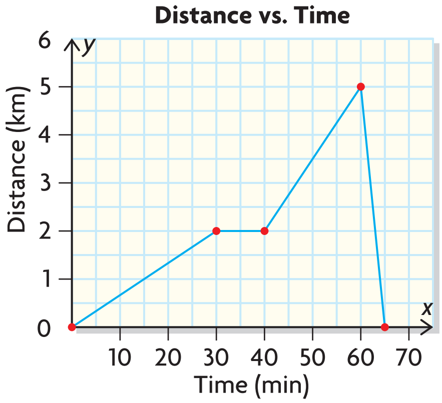
The price of a turkey is 29¢/kg less at the supermarket.

Your Turn

Describe how you could use a graph to compare the price of a turkey from the butcher and from the supermarket.

EXAMPLE 2 Connecting the slope of a line segment to a rate

Describe a scenario that could be represented by this graph. Compare the rates shown, and discuss why the rates may have changed.



Gilles's Solution

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\frac{\Delta d}{\Delta t} = \frac{d_2 - d_1}{t_2 - t_1}$$

To choose a reasonable scenario, I decided to calculate the slope of each line segment. This meant dividing the change in distance by the change in time.

The first line segment has endpoints (0, 0) and (30, 2).

$$\text{Slope}_1 = \frac{2 - 0}{30 - 0}$$

$$\text{Slope}_1 = \frac{2}{30}\text{ km/min}$$

$$\left(\frac{2\text{ km}}{30\ \cancel{\text{min}}}\right)\left(\frac{60\ \cancel{\text{min}}}{1\text{ h}}\right) = 4\text{ km/h}$$

The second line segment is horizontal.

$$\text{Slope}_2 = 0\text{ km/h}$$

The slope represents the rate of change of distance over time, in kilometres per minute, for each line segment. This speed is a unit rate, since the numerical value of the second term is equal to 1.

I converted each rate in kilometres per minute to kilometres per hour to help me describe the scenario. The speed represented by the first line segment is 4 km/h.

The third line segment has endpoints (40, 2) and (60, 5).

$$\text{Slope}_3 = \frac{5 - 2}{60 - 40}$$

$$\text{Slope}_3 = \frac{3}{20}\text{ km/min}$$

$$\left(\frac{3\text{ km}}{20\ \cancel{\text{min}}}\right)\left(\frac{60\ \cancel{\text{min}}}{1\text{ h}}\right) = 9\text{ km/h}$$

The fourth line segment has endpoints (60, 5) and (65, 0).

$$\text{Slope}_4 = \frac{0 - 5}{65 - 60}$$

$$\text{Slope}_4 = \frac{-5}{5}\text{ or } -1\text{ km/min}$$

$$\left(\frac{-1\text{ km}}{1\ \cancel{\text{min}}}\right)\left(\frac{60\ \cancel{\text{min}}}{1\text{ h}}\right) = -60\text{ km/h}$$

The graph could represent the distance that a person travelled and the time for the trip. Each line segment on the graph could represent a different part of the trip.

I decided that the rates of change were close to the rates at which a person could walk, run, and drive. I made up a story to match the graph and the rates of change that I calculated

The first line segment represents a person walking at a rate of 4 km/h for 30 min to get from home to a variety store.

4 km/h is a reasonable rate at which a person could walk. The graph shows that the distance is increasing during this period.

The second line segment represents the person stopping at the store for 10 min to buy something or talk to someone.

This makes sense, since the distance travelled did not change between 30 min and 40 min.

The third line segment represents the person jogging at a rate of 9 km/h for 20 min to get to a friend's house.

It's difficult to walk at 9 km/h, so I decided that the person must be jogging. The distance was increasing faster during this period.

The fourth line segment represents the person travelling at the greatest rate, 60 km/h, for an additional 5 min. This is because the person received a phone call from home, to say that dinner is ready. The person gets a ride home from a friend.

The rate of change is negative. This means that the person's distance, relative to the starting point, is decreasing. The person must be travelling back to the starting position. A person can't walk at 60 km/h, so I decided that the person must be travelling by car.

EXAMPLE 3 Solving a problem involving rates

When making a decision about buying a vehicle, fuel efficiency is often an important factor.

The gas tank of Mario's new car has a capacity of 55 L. The owner's manual claims that the fuel efficiency of Mario's car is 7.6 L/100 km on the highway. Before Mario's first big highway trip, he set his trip meter to 0 km so he could keep track of the total distance he drove. He started with the gas tank full. Each time he stopped to fill up the tank, he recorded the distance he had driven and the amount of gas he purchased.

Fill-up	Total Distance Driven (km)	Quantity of Gas Purchased (L)
1	645	48.0
2	1037	32.1

On which leg of Mario's trip was his fuel efficiency the best?

Katrina's Solution: Using a graph

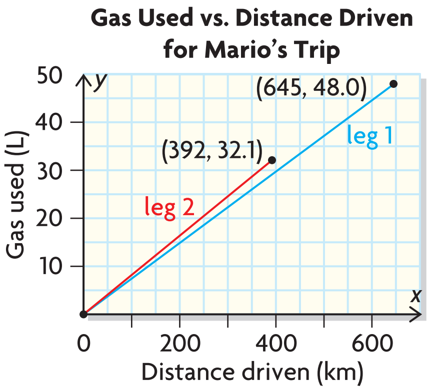
Distance driven on leg 1 = 645 km

Distance driven on leg 2 = 1037 − 645
Distance driven on leg 2 = 392 km

First, I needed to determine how far Mario drove on each leg of the trip. Since the trip meter was initially set to 0 km, I knew that he drove 645 km on the first leg. I had to subtract the first trip meter reading from the second trip meter reading to determine the distance he drove on the second leg.

Gas used on leg 1 = 48.0 L
Gas used on leg 2 = 32.1 L

Since Mario began each leg with a full tank, the amount of gas he purchased at each fill-up tells me the amount of gas he used on each leg of the trip.



I created a graph of Gas used versus Distance driven by plotting the ordered pairs (Distance driven, Gas used). I used distance driven on the horizontal axis, because I noticed that distance is the second term in the manufacturer's fuel efficiency. The slope of the line segment for each leg of the trip represents the change in fuel use over distance.

The blue line segment is less steep than the red line segment, so its slope is less. This means the fuel efficiency of the car was better on the first leg of the trip.

The fuel efficiency is better when you use less gas to drive the same distance, so I knew I was looking for the line segment that was less steep.

Grant's Solution: Using unit rates

Leg	Distance Driven (km)	Gas Used (L)
1	$645 - 0 = 645$	48.0
2	$1037 - 645 = 392$	32.1

I used a table to organize the information I needed to determine the fuel efficiency on each leg of the trip as a unit rate, in litres per kilometre. For each leg of the trip, I had to determine the distance that Mario drove and the amount of gas that was used.

First leg:

$$\text{Fuel efficiency} = \frac{\text{Gas used}}{\text{Distance driven}}$$

$$\text{Fuel efficiency} = \frac{48.0\text{ L}}{645\text{ km}}$$

$$\text{Fuel efficiency} = 0.074\ldots\text{ L/km}$$

I calculated the car's fuel efficiency on each leg of the trip by dividing the gas used by the distance driven.

Second leg:

$$\text{Fuel efficiency} = \frac{\text{Gas used}}{\text{Distance driven}}$$

$$\text{Fuel efficiency} = \frac{32.1\text{ L}}{392\text{ km}}$$

$$\text{Fuel efficiency} = 0.081\ldots\text{ L/km}$$

The fuel efficiency of the car was better on the first leg of the trip.

The fuel efficiency is better when you use less gas to drive the same distance.

Your Turn

Did the car achieve the manufacturer's fuel efficiency rating of $\frac{7.6\text{ L}}{100\text{ km}}$ on either leg of the trip? Explain.

In Summary

Key Idea

- Rates can be represented in a variety of ways. The representation you choose should depend on your purpose.

Need to Know

- You can compare rates by writing them as
  - rates with the same units, with second terms that are numerically the same.
  - unit rates, where the numerical values of the second terms are equal to 1.
- When comparing rates, it is helpful to round the values. This enables you to do mental math and express each rate as an approximate unit rate.
- In a graph that shows the relationship between two quantities, the slope of a line segment represents the average rate of change for these quantities.
- The slope of a line segment that represents a rate of change is a unit rate.

CHECK Your Understanding

1. Compare the following situations, and determine the lower rate.

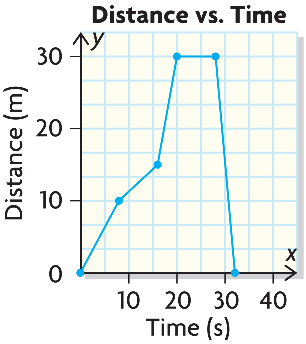
a) At store A, 8 kg of cheddar cheese costs $68.
At store B, 12 kg of cheddar cheese costs $88.20.

b) At gas station A, 44 L of fuel costs $41.36.
At gas station B, 32 L of fuel costs $31.36.

2. Compare the following situations, and determine the greater rate.

a) It takes 4 h 15 min to drain tank A, which holds 300 L of water.
It takes 2 h 10 min to drain tank B, which holds 150 L of water.

b) Person A runs 400 m in 1 min 15 s.
Person B runs 1 km in 5 min 20 s.



3. The graph to the left shows how an all-terrain vehicle (ATV) travels over time.

a) Over which interval of time is the ATV travelling the slowest? Over which interval is it travelling the fastest?

b) When does the ATV start to return to its starting position? When does it get there?

c) What does a zero slope mean in the context of this graph?

PRACTISING

4. Apple juice is sold in 1 L bottles and 200 mL boxes. A 1 L bottle sells for \$1.75, and fifteen 200 mL boxes sell for \$4.99.
   a) Determine the unit rate, in dollars per millilitre, for each size.
   b) Which size has the lower cost per millilitre?

5. The list price for a 925 mL container of paint is \$20.09. A 3.54 L container of the same paint costs \$52.99. Which container has the lower unit cost?

6. When Rupi goes to her aerobics class, she can burn 140 Cal in 20 min. When she plays hockey for 1.5 h, she can burn 720 Cal. Which activity burns Calories at a greater rate?

7. For each of the following, compare the two rates and determine the lower rate.
   a) whole chickens: \$3.61/kg or 10 lb for \$17.40
   b) jogging speeds: 6 mph or 2 km in 10 min
   c) fuel efficiency: 10.6 L/100 km or 35.1 L of fuel needed to travel 450 km
   d) driving speeds: 30 m/s or 100 km/h

Communication | ***Tip***

1 kg $\doteq$ 2.2 lb
1 km $\doteq$ 0.6 mi (miles)

8. Jay can buy a 25 lb bag of bird seed for \$21.30 from the Farmers Co-op. The pet store in town sells an 18 kg bag for \$24.69. At which store can Jay buy bird seed at a lower cost? Explain how you know.

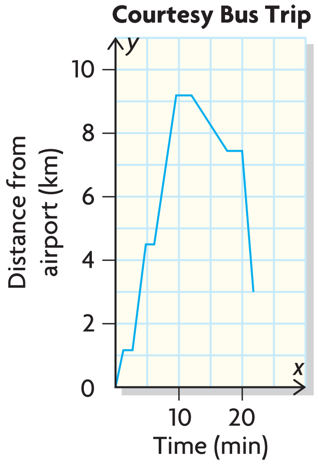
9. Shelley has two choices for a long-distance telephone plan:
- her telephone company, which charges 4¢/min
- a device that plugs into her Internet modem, which costs \$19.95 with an additional charge of 1.5¢/min

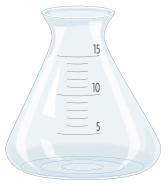
Shelley makes, on average, 50 min of long-distance calls per month. Which option would be cheaper on an annual basis? Justify your decision.

10. On Monday, a crew paved 10 km of road in 8 h. On Tuesday, the crew paved 15 km in 10 h. Draw a graph to compare the crew's daily paving rates.

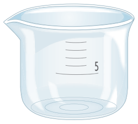
11. Draw a graph that shows how, over one day, the outdoor temperature starts at 24 °C, decreases at a rate of 1.5 °C/h for 5 h, remains constant for 2 h, and then increases by 0.75 °C/h for 4 h.

12. A hotel shuttle bus takes David from the airport to his hotel. Use the distance versus time graph to the right to create a story that describes David's bus trip.

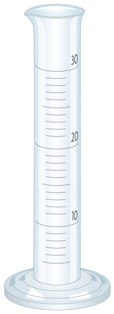




flask



beaker

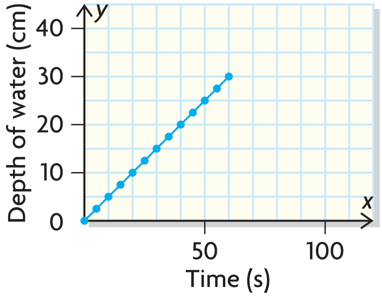


graduated cylinder

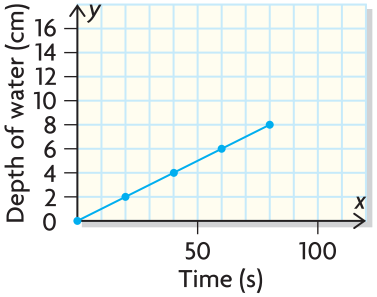
drinking glass

13. Suppose that tap water, flowing from a faucet at a constant rate, is used to fill these containers. Match each of the following graphs with the appropriate container. Justify your choices.

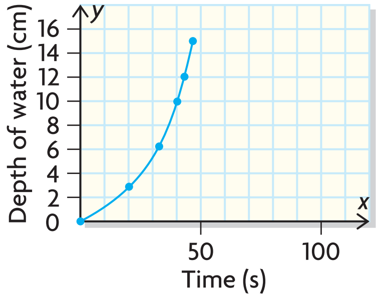
a)



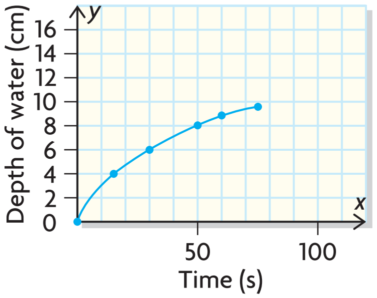
c)



b)



d)



14. The following table shows the amount of greenhouse gases emitted by fossil fuel production in Canada from 1990 to 2006. During which period was the amount emitted increasing at the greatest rate? Justify your decision.

Year	Greenhouse Gases from Fossil Fuel Production (megatonnes)
1990	103
1995	127
2000	151
2003	161
2006	158

15. At the 2002 Olympics, speed skater Cindy Klassen of Winnipeg, Manitoba, finished out of the medals. Four years later, at the 2006 Olympics, she won five medals in women's speed skating, including gold in the 1500 m race. Compare her speeds in the 1500 m race in 2002 and 2006. In which portion of the race did her speed differ the most?

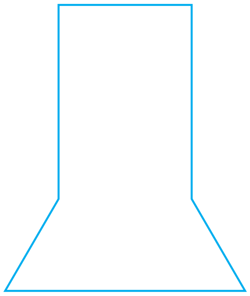
Games	Age	City	Rank	Time	300 m	700 m	1100 m
2002 Winter	22	Salt Lake	4	1:55.59	25.65	54.17	1:24.09
2006 Winter	26	Torino	1	1:55.27	25.42	53.83	1:23.50

Closing

16. a) When comparing rates, when is an estimate sufficient and when is a precise solution needed? Use specific examples to support your answer.

b) When comparing rates, when is a graphing strategy a good approach and when is a numerical strategy better? Use specific examples to support your answer.

Extending



17. Water is poured into a container like the one to the right at a constant rate. Sketch a graph of depth versus time to represent this situation.

18. Scientists estimate that the processing power of the human brain is equivalent to about 100 million MIPS (million computer instructions per second).

a) In 2005, a typical home computer could process about 7000 MIPS. About how many of these computers were equivalent to the processing power of the brain?

b) Determine the processing power of a computer you use. How many of these computers are equivalent to the processing power of the brain?

History Connection

Ivy Granstrom

Ivy Granstrom was born in 1911 in Glace Bay, Nova Scotia, but lived most of her life in Vancouver, British Columbia. She was blind from birth. For 76 consecutive years, she participated in the annual English Bay Polar Bear swim and was affectionately known as the Queen of the Polar Bears. When she was 64 years old, she was struck by a car. Doctors told her that she would always need a wheelchair. However, she began her own rehab program, walking, jogging, and then running. Soon after, she participated in blind sports competitions for the visually impaired. She established herself as one of world's fastest runners in the Masters Division, setting 25 world records for athletes aged 60 years and older, competing against sighted athletes. She raced until 2001 with Paul Hoeberigs, who ran behind, tethered by a cord and calling out directions. Ivy was a Sports B.C. Athlete of the Year in 1982, inducted into the Terry Fox Hall of Fame in 2001, and made a Member of the Order of Canada in 1989. In 2009, she still held the world record in the 1500 m race for women aged 85 and over. Her time of 10:33.40 was established in 1997. She passed away in 2004.

A. If Granstrom ran the mile race at her world record rate for the 1500 m race, could she break the record, which was 11:03.11 in 2009? Justify your answer.

8.2

Solving Problems That Involve Rates

YOU WILL NEED

- calculator
- graph paper
- ruler

EXPLORE...

- A car travels at 80 km/h. What other rates, expressed using different units, could be used to describe the speed of the car? What would be some of the advantages of using these other rates?

GOAL

Analyze and solve problems that involve rates.

LEARN ABOUT the Math

Jeff lives in a town near the Canada–U.S. border. The gas tank of his truck holds about 90 L. He can either buy gas in his town at \$1.06/L or travel across the border into the United States to fill up at \$2.86 U.S./gal.

? Which option makes the most sense economically?

EXAMPLE 1 Solving a problem that involves multiple rates

Jeff's Solution

The cost to fill up at a gas station in Canada, D, is

$$D = (90\ \cancel{\text{L}})\left(\frac{\$1.06\text{ Cdn}}{1\ \cancel{\text{L}}}\right)$$

$$D = \$95.40\text{ Cdn}$$

It will cost \$95.40 to fill up in Canada.

First, I determined the cost to fill up at a gas station in Canada.

Converting 90.0 L into U.S. gallons, G, is

$$G = (90\ \cancel{\text{L}})\left(\frac{1\text{ gal}}{3.79\ \cancel{\text{L}}}\right)$$

$$G = 23.746...\text{ gal}$$

I needed to convert the volume that my gas tank holds in litres into U.S. gallons. I know that 1 U.S. gallon is equivalent to 3.79 L.

The cost in U.S. dollars, U, for about 23.7 U.S. gal is

$$U = (23.746...\ \cancel{\text{gal}})\left(\frac{\$2.86\text{ U.S.}}{1\ \cancel{\text{gal}}}\right)$$

$$U = \$67.915...\text{ U.S.}$$

I determined the cost to fill up in U.S. dollars. I needed to convert U.S. dollars into Canadian dollars. The exchange rate today is \$1 U.S./\$1.02 Cdn.

The cost in Canadian dollars, C, for about 23.7 U.S. gal is

$$C = (\$67.915...\ \cancel{\text{U.S.}})\left(\frac{\$1.02\text{ Cdn}}{1\ \cancel{\text{U.S.}}}\right)$$

$$C = \$69.273...\text{ Cdn}$$

It will cost \$69.27 Cdn to fill up in the United States.

Difference in cost $= \$95.40 - \$69.273...$

Difference in cost $= \$26.13...$

Today, it is more economical for me to fill up in the United States. I will save about \$26.

How much I save depends on the price of gas at each station.

Reflecting

A. What other factors will affect Jeff's savings each time he considers where to fill up?

B. Jeff has only considered the cost to fill up his truck. What other factors should he consider when deciding where he will buy gas?

C. Jeff thinks that saving less than $10 is not worth his time. If he had half a tank of gas in his truck, would it be worthwhile for him to fill up in the United States today? Justify your answer.

APPLY the Math

EXAMPLE 2 Connecting rates to contextual situations

Describe a situation in which each unit rate might be used. Identify and explain factors that could influence the unit rate in this situation.

a) 0.05 mg/kg **b)** 98.5¢/L **c)** 7.2 MBps

Mangat's Solution

a) 0.05 mg/kg could be the rate at which a certain type of medicine must be administered. This rate means that 0.05 mg of medication is needed for each kilogram of a patient's mass. The type of medication used could influence the quantity administered per kilogram of body mass.

I needed a situation in which a very small mass of a substance (milligrams) is related to a kilogram. I knew that medicine is prescribed according the mass of a patient, which can be measured in kilograms.

b) 98.5¢/L could represent the rate at which consumers pay for 1 L of gasoline.
This rate could be influenced by the type of gas chosen. It could also be influenced by the current cost of gasoline per barrel, which could be affected by war, weather, time of year, holidays, and supply and demand.

I needed a situation in which a cost in cents is related to a volume in litres. Gasoline is sold in litres.

c) 7.2 MBps could be the rate at which information is transferred over a computer network.
This rate could be influenced by
- the type of network (wireless versus wired).
- the type/quality of the network card.
- the type of router used.

I needed a situation in which megabytes (MB) are related to time in seconds. Data transfer can be measured this way.

Your Turn

Think of two other unit rates that you are familiar with. State one or two factors that could influence these rates.

EXAMPLE 3 Reasoning to solve a rate problem

Paula is asked to order snacks for an office meeting of 180 people. She decides to order dessert squares, which come in boxes of 24. She estimates that she will need 2.5 squares/person. How many boxes should she buy?

Mila's Solution: Calculating using unit analysis

Formula to describe the snack order:

$$\text{Number of boxes} = \left(\frac{1\text{ box}}{\text{Number of squares}}\right)\left(\frac{\text{Number of squares eaten}}{1\text{ person}}\right)(\text{Number of people})$$

$$\text{Number of boxes} = \left(\frac{1\text{ box}}{24\ \cancel{\text{squares}}}\right)\left(\frac{2.5\ \cancel{\text{squares}}}{1\ \cancel{\text{person}}}\right)(180\ \cancel{\text{persons}})$$

Paula estimated that each person would eat about 2.5 squares.

$$\text{Number of boxes} = 18.75$$

Paula should buy 19 boxes.

I rounded up my answer to the nearest number of boxes.

Joe's Solution: Estimating using proportional reasoning

There are about 25 squares in each box.
If each person eats 2.5 squares, then Paula needs one box for every 10 people.
There are 18 groups of 10 in 180.

I decided to estimate. I knew that each person will eat about 2.5 squares. Estimating 25 squares/box made the numbers easier to work with, using mental math.

Paula needs to buy at least 18 boxes.
She should order 19 boxes to be safe.

I know that I underestimated, since I estimated 25 squares/box and there are only 24 squares/box.

Your Turn

If each person at the meeting eats about 1.5 squares on average, how many boxes of squares will be left over?

EXAMPLE 4 Solving a problem that involves different rates

Amelia walks briskly, at 6 km/h. When she walks at this rate for 2 h, she burns 454 Cal. Bruce walks at a slower rate, 4 km/h, burning 62 Cal in 30 min. If Amelia walks for 3 h, how much longer will Bruce have to walk in order to burn the same amount of Calories?

April's Solution: Using a function

The amount of Calories that Amelia burns each hour when she walks at 6 km/h, A, is

$$A = \frac{454\text{ Cal}}{2\text{ h}}$$

$$A = 227\text{ Cal/h}$$

If $A(t)$ represents the amount of Calories that Amelia burns and t represents the time in hours, then

$A(t) = 227t$

I assumed that Amelia walks at a constant rate, so I could use a linear function to represent the relation between Calories burned and time.

For 3 h,

$$A(3) = 227(3)$$
$$A(3) = 681 \text{ Cal}$$

Amelia burns 681 Cal in 3 h.

The amount of Calories that Bruce burns each hour when he walks at 4 km/h, B, is

$$B = \frac{62 \text{ Cal}}{0.5 \text{ h}}$$
$$B = 124 \text{ Cal/h}$$

If $B(t)$ represents the amount of Calories that Bruce burns and t represents the time in hours, then

$B(t) = 124t$

I assumed that Bruce also walks at a constant rate.

For 681 Cal,

$$\begin{aligned} B(t) &= 681 \text{ Cal} \\ 681 \text{ Cal} &= (124 \text{ Cal/h})t \\ 5.491\ldots \text{ h} &= t \end{aligned}$$

I needed to know how long it takes Bruce to burn 681 Cal, so I substituted 681 Cal for B(t).

Bruce will need to walk for about 5.5 h to burn the same amount of Calories, that Amelia burns in 3 h. Bruce will need to walk an additional 2.5 h.

I rounded the time to the nearest tenth of an hour.

Joanna's Solution: Using equivalent ratios

The rate for Calories burned is Cal/h. When Amelia walks 6 km/h for 2 h, she burns 454 Cal. When she walks for 3 h, she burns x Calories. These rates are equivalent.

I assumed that Amelia walks at a constant rate, so the rate at which she burns calories will also be constant.

$$\frac{454 \text{ Cal}}{2 \text{ h}} = \frac{x}{3 \text{ h}}$$
$$3 \cancel{\text{h}}\left(\frac{454 \text{ Cal}}{2 \cancel{\text{h}}}\right) = x$$
$$681 \text{ Cal} = x$$

I wrote an equation using equivalent rates to determine the amount, in Calories, that Amelia burns in 3 h. The units in the ratios are the same, so I am confident that my answer will be in the correct units, Calories.

Bruce burns 62 Cal in 0.5 h. He must burn 681 Cal in t hours.

$$\frac{62 \text{ Cal}}{0.5 \text{ h}} = \frac{681 \text{ Cal}}{t}$$
$$62t = 340.5 \text{ h}$$
$$t = 5.491\ldots \text{ h}$$

I wrote an equation using a pair of equivalent rates, where 681 Cal must be burned by Bruce to match Amelia. Then I solved for t.

Converting 0.491... h into minutes, M, is

$$M = (0.491\ldots\ \cancel{h})\left(\frac{60\text{ min}}{1\ \cancel{h}}\right)$$

$$M = 29.516\ldots\text{ min}$$

Bruce will need to walk for about 5 h 30 min to burn the same amount of Calories that Amelia burns in 3 h.

Bruce will need to walk about 2 h 30 min longer to burn the same amount of Calories.

Your Turn

If Bruce walks for 2 h, how long does Amelia need to walk to burn the same amount, in Calories, as Bruce? Round your answer to the nearest minute.

In Summary

Key Idea

- When you are given a rate problem that involves an unknown, you can solve the problem using a variety of strategies.

Need to Know

- Often, a problem that involves rates can be solved by writing an equation that involves a pair of equivalent ratios. To be equivalent ratios, the units in the numerators of the two ratios must be the same, and the units in the denominators must be the same. Paying attention to the units in each term of the ratios will help you write the equation correctly.
- A multiplication strategy can be used to solve many rate problems, such as problems that require conversions between units. Including the units with each term in the product and using unit elimination helps you verify that your product is correct.
- When a rate of change is constant, writing a linear function to represent the situation may be useful when solving problems.

CHECK Your Understanding

1. a) 50 L of oil costs \$163. How much oil, to the nearest litre, could you buy for \$30?

b) It takes 3 min 25 s to fill a 75 L gas tank. How long, to the nearest minute, will it take to fill a 55 L gas tank?

c) 8 kg of beef costs \$68.00. How much will it cost, to the nearest cent, for 1.5 kg of beef?

d) The adult dosage of an antibiotic medicine is 25 mL/80 kg. How much medicine is needed for a person with a mass of 95 kg?

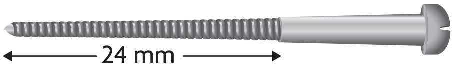
2. Two competing stores have 350 mL cans of pop on sale this week. Supersaver is selling a case of 24 cans for \$5.99. Gord the Grocer is selling cans of the same pop in cases of 12, with three cases for \$9.99.

a) Which store is selling soft drinks at the lower price per can?

b) Besides price, what other factors should be considered when determining which store offers the better buy for a consumer?

PRACTISING

3. A screw has 32 turns over a distance of 24 mm of thread.



Another screw, with the same pattern, has 42 mm of thread. How many turns does it have?

4. The Wildcats won 12 of their first 20 games. At this rate, predict how many games they will win during the 30-game season.

5. Mario borrowed \$1000 and paid \$40 simple interest. If he borrowed the money for eight months, what interest rate was he charged?

6. Describe a situation in which each rate might be used. Identify any factors that could influence the rate in this situation.

a) \$7.23/kg

b) 20 mL/90 kg

c) \$1.08/100 g

d) −1.5 °C/km

e) 20 g/L

f) \$4.99/ft^2

7. Basic units of data are transferred by a particular computer at 12 MB (megabytes) every 2 s. How long will it take this computer to transfer 1.5 GB (gigabytes) of data? (1 GB is equivalent to 1024 MB.)

8. Melanie wants to defrost a frozen roast, which weighs 2.68 kg, in her microwave. To find out how much time she needs, she looks in a cookbook. She reads that 2 lb of meat takes 15 min to defrost. How long, to the nearest minute, should she set the timer for?

9. A nurse administers a vaccine that comes in a 10 mL bottle. The adult dosage is 0.5 cc (1 cc = 1 mL). How many adults can the nurse vaccinate before the bottle is empty?

10. Tonya works 50 h every three weeks. At this rate, how many hours will she work in one year? Explain how you could solve this problem using two different strategies.

11. Chris and her friend Elena drove from Vancouver to Yellowknife for a reunion. They took turns driving, so they only needed to stop for gas or food. They drove the 2359 km distance in 36 h 12 min.

a) Determine their average speed to the nearest tenth of a kilometre per hour.

b) They used 231.2 L of fuel. Determine their average fuel consumption per 100 km.

c) Chris and Elena spent $252.05 on fuel. What was the average cost of a litre of gas?

12. Manpret has taken a job as a nurse in the community health centre in Tuktoyaktuk, Northwest Territories. She plans to ship her car, furniture, and personal effects to Tuktoyaktuk by barge from Vancouver. She has found these shipping rates online:

- light-duty vehicles: $0.2015/lb
- furniture and personal effects: $0.2734/lb

Manpret knows that her car has a mass of 1250 kg. She estimates that she has roughly 550 lb of furniture and personal effects. Calculate her cost to ship these items to her destination.

Communication | ***Tip***

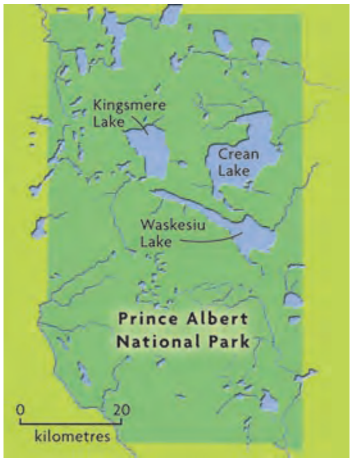
1 kg ≐ 2.2 lb

13. Emma runs a kennel near Wild Horse, Alberta. She has decided to purchase dog food from a U.S. supplier. The supplier sells 40 lb bags for $38.95 U.S. The exchange rate is $1 U.S. for $1.05 Cdn on the day that she orders the food.

a) How much, in Canadian dollars, does Emma spend to buy 20 bags of dog food?

b) Each dog eats about 4 kg/week and Emma boards an average of 12 dogs per day. Will the 20 bags of dog food last two months? Explain.

c) What other factors should she have considered before she ordered from this supplier?



14. The map to the left shows Prince Albert National Park in Saskatchewan. The scale of the map is 1.3 cm to 20 km.

a) Estimate the area of the park in hectares. One hectare (1 ha) is equivalent to 10 000 m^2.

b) The annual cost to monitor and fight forest fires in this region is about $48/ha. Estimate the annual fire management expenditure for the park.

15. Paula wants to buy bottled water.

- Store A, located 12 km from her home, is selling 500 mL bottles in a case of 24 for $4.99.
- Store B, located 20 km from her home, is selling 330 mL bottles in a case of 24 for $3.49.
- It costs Paula $0.14/km to run her car.

Which store would you recommend for Paula to buy her water? Explain.

16. A cargo jet leaves an airport that is 2000 ft above sea level at 6:30 a.m. The jet climbs steadily to a cruising altitude of 37 000 ft, at a rate of 7000 ft/min. After cruising at this altitude for 40 min, the jet descends steadily at a rate of 3500 ft/min to an airport that is 5500 ft above sea level. What time does the jet land?

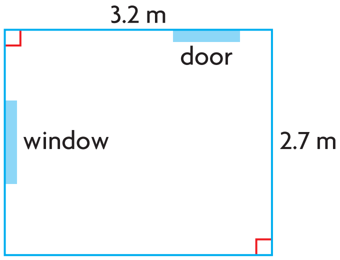
17. The low temperature for a certain day was recorded as −5.3 °C at 3:30 a.m. The temperature then rose steadily until the high temperature was recorded as 11.8 °C at 5:45 p.m. A weather forecaster predicted the same temperature increase rate for the next day, from a low of −7 °C at 3 a.m. Estimate the temperature at 7 a.m. the next day.

Closing

18. A particular type of paint can be purchased at two local stores. Bren's Interior Design sells the 870 mL size for $7.99, while Home Suppliers sells the 3.7 L size for $27.99. This type of paint will cover an area of 10 m^2/L. Suppose that you want to paint a room that is 2.4 m high and has the dimensions shown to the right. One wall has a door that measures 80 cm by 205 cm. Another wall has a window that measures 100 cm by 130 cm.

a) Based on cost, at which store should you buy the paint?

b) In addition to cost, what other factors should you consider when deciding where to buy the paint?

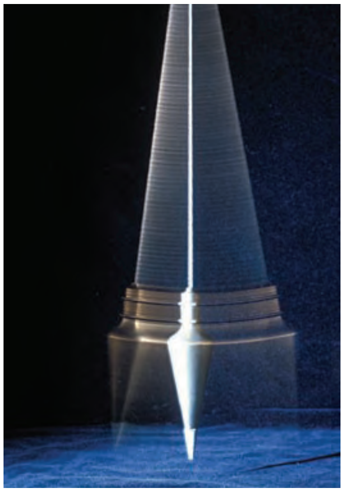


Extending

19. A pendulum is pulled to the left. When released, it swings from left to right, but never returns to its initial position. The time required for one complete oscillation is called the *period* of the pendulum. The time, T, in seconds, for one period of the pendulum is given by the equation

$$T = 2\pi\sqrt{\frac{L}{9.8}},$$

where L is the length of the pendulum in metres. How many periods will a 2 m pendulum complete over 1 h?



20. A new saltwater pool is being filled by four different pumps, which pump water from a nearby ocean into the pool. The first pump can fill the entire pool with water in two days. The second pump requires three days, and the third pump requires four days. The fourth pump needs only 6 h. How long will it take to fill the pool if all four pumps are used?

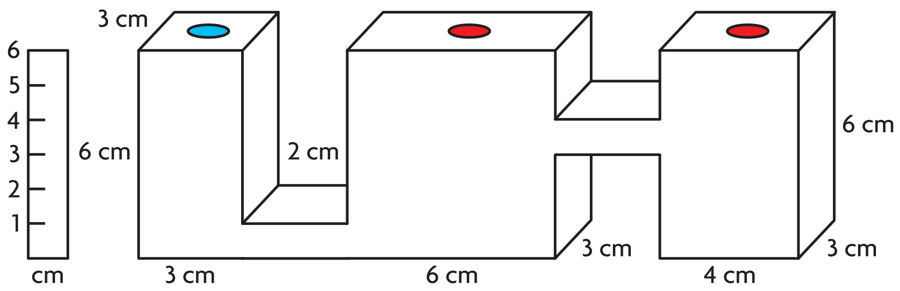
Applying Problem-Solving Strategies

Analyzing a Rate Puzzle

Filling irregular shaped containers with water at a constant rate can produce some interesting results.

The Puzzle

A container is constructed from a connected sequence of rectangular prisms as shown below.



Each connecting piece has dimensions of 1 cm by 2 cm by 3 cm. Water is dripping into the hole on the left (marked in blue) at a constant rate of 1 cm^3/min. The marks to the left of the container measure the height, in centimetres, of the water in the container as it fills.

A. Will one prism be filled before the others? Explain.

B. Determine the time needed for the water to reach each of the height marks indicated to the left of the container.

C. Use your times to plot a graph of water height versus time.

D. Between which two height markers did the water level rise at the slowest rate? Explain how you know.

The Strategy

E. Describe the strategy you used to determine the information needed to create your graph.

Modifying the Puzzle

F. There are other holes in the container, indicated in red. If the container were filled at the same rate through either of those holes, would the length of time needed to fill the container change? Explain.

G. Would your graph change if you filled the container through a different hole? Explain.

8 Mid-Chapter Review

FREQUENTLY ASKED Questions

Study | Aid

- See Lesson 8.1.
- Try Mid-Chapter Review Questions 1 to 5.

Q: **How do you compare rates? When is one strategy more effective than another?**

A: Here are three strategies you can use to compare rates:

- It is often effective to express the rates as unit rates using the same units.

  For example, organic cashews may be sold for $18.95/kg at one store and $9.49/lb at another store. To determine which rate, or price, is less expensive, you can convert the rate in kilograms to a rate in pounds and then calculate the equivalent unit rate to make a proper comparison. (Alternatively, you could convert the rate in pounds to a rate in kilograms. The choice of which rate to convert might be affected by other comparisons that you need to make to solve a problem.)

$9.49/lb	$18.95/kg
	1 kg : 2.2 lb
	$\frac{\$18.95}{1\ \cancel{\text{kg}}}\left(\frac{1\ \cancel{\text{kg}}}{2.2\text{ lb}}\right) \doteq \$8.61/\text{lb}$
	$18.95/kg is less expensive.

  This strategy is effective when you need to know and use the numerical value of each rate.

- On a graph of a relation, the slope of a line that joins two points is equivalent to the average rate of change.

  For example, car A travels 50 m in 4 s, and car B travels 40 m in 4 s. If these data are plotted on a graph, it is clear that car A is travelling at a greater rate than car B, because the blue line is steeper than the red line.

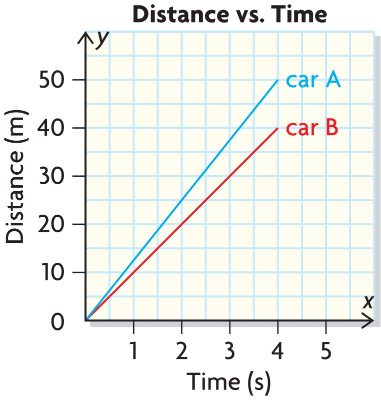
  This strategy is effective when you need to know which rate is greater or lesser, but you do not need to know the numerical values. However, the numerical values could be determined by calculating the slopes of the lines:

$$\text{Slope} = \frac{\Delta y}{\Delta x}$$

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{Rate of change for car A} = \frac{50 - 0}{4 - 0} \text{ or } 12.5 \text{ m/s}$$

$$\text{Rate of change for car B} = \frac{40 - 0}{4 - 0} \text{ or } 10.0 \text{ m/s}$$



- Rates can also be compared by writing them as equivalent rates, with the second terms numerically the same.
  For example, suppose that you burn 320 Cal in 20 min of spin class and 210 Cal in 15 min of jogging. You can compare these rates by determining the amount of Calories burned in an hour by doing each activity, using the fact that there are 60 min in an hour.

Spinning:	Jogging:
$\frac{320\text{ Cal}}{20\text{ }\cancel{\text{min}}}\left(\frac{60\text{ }\cancel{\text{min}}}{1\text{ h}}\right) = 960\text{ Cal/h}$	$\frac{210\text{ Cal}}{15\text{ }\cancel{\text{min}}}\left(\frac{60\text{ }\cancel{\text{min}}}{1\text{ h}}\right) = 840\text{ Cal/h}$

Spinning burns Calories at a greater rate.

Study Aid

- See Lesson 8.2, Example 1.
- Try Mid-Chapter Review Question 6.

Q: When you are comparison shopping, what factors, other than unit price, should you consider?

A: The factors to consider will depend on the situation.

For example, if you want to buy a pair of jeans and you know the prices at two different stores, you might also consider

- the distance to each store and the time you will need to get there.
- the cost of fares you will pay or gas you will use to travel there and back.
- how busy each store will be.
- the exchange rate on the dollar, if one or both stores are located in the United States.

Study Aid

- See Lesson 8.2, Example 1, Example 3 (Mila's Solution), and Example 4 (Joanna's Solution).
- Try Mid-Chapter Review Questions 7 and 8.

Q: Why is analyzing the units in a rate problem a useful strategy?

A1: Often, a problem that involves rates can be solved by writing an equation. The equation you write will involve a pair of equivalent ratios. In this kind of equation, the units in the numerators of the two ratios must be the same and the units in the denominators must be the same. Paying attention to the units in each term of these ratios will help you write the equation correctly.

A2: Sometimes, a rate problem can be solved by using a multiplication strategy. When you use this strategy, including the units with each term in the product will help you verify that you have multiplied the quantities correctly. The units should cancel to leave you with the correct units for your answer.

PRACTISING

Lesson 8.1

1. Carol can key at the rate of 65 words/min. Jed can key 290 words in 5 min. Who is faster? Explain how you know.

2. Harry filled the 75 L gas tank of his pickup truck for $73.88. Stan paid 95¢/L to fill up his truck. Who paid less per litre for fuel?

3. For each of the following, compare the two rates and determine the lower rate.

a) Calories burned: 300 Cal/h or 4 Cal/min
b) water usage: 30 L/day or 245 L/week
c) ground beef: $8.40/kg or $3.99/lb
d) cycling speeds: 2 miles in 5 min or 5 km in 20 min

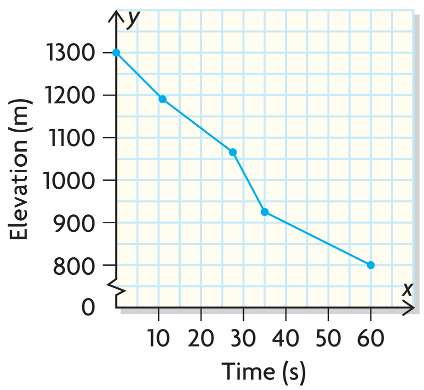
4. a) Draw a graph to show Lyn's body temperature, based on this description:

- rising at a constant rate from 98.6 °C to 102 °C over a period of 3 h
- remaining at 102 °C for 2 h
- falling back to 98.6 °C over a period of 5 h

b) During which interval of time was the rate of change in her body temperature the greatest?

5. The following graph shows elevation versus time for a skier who descended a mountain.

Elevation vs. Time



a) During which interval of time was the skier's speed the greatest? Explain.

b) During which interval of time was the skier's speed the least? Explain.

c) Estimate the skier's speeds for the intervals you identified in parts a) and b).

d) What was the skier's average speed over the entire run?

Lesson 8.2

6. Martin is shopping for a new MP3 player. The one he wants is on sale for $119.95 at Giant Electronics, located in his town. He has found the same MP3 player for $105.99 on the Internet, on the website for a U.S. store. Today's exchange rate is $1 U.S. = $1.08 Cdn.

a) Determine which store has the lower price in Canadian dollars.

b) What factors, besides the list price, should Martin consider before he makes the purchase?

7. An airplane travels 300 miles in 36 min. At this rate, how far will it travel in 2 h?

8. Sam bought a used fishing boat in the United States and brought it back to Canada. According to the literature that came with the boat, the gas tank holds 25 gal. The marina where Sam docks his boat sells gas for $1.08/L. Determine the cost to fill the gas tank at this marina. (The conversion rate is 1 U.S. gal/3.785 L.)

9. Hicham El Guerrouj of Morocco ran 1500 m in 00:03:26.00 in Rome, Italy, in July 1998. Just under a year later, he ran the mile in 00:03:43.13 on the same track.

a) Determine his average speed in each race.

b) Compare the distances run and his average speeds in both races.

c) Discuss some factors that may have led to his average speeds being different in these two races.

8.3

Scale Diagrams

YOU WILL NEED

- calculator
- grid paper
- ruler
- protractor

EXPLORE...

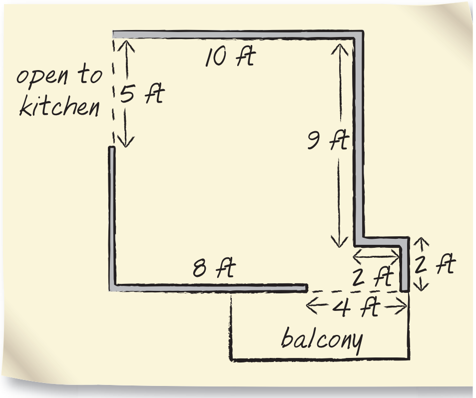
- Construct a scale drawing that models an airplane flying N20°E at 160 km/h for 2.5 h. What factors must you keep in mind?

GOAL

Understand and use scale diagrams involving 2-D shapes.

INVESTIGATE the Math

Maxine is moving into a new apartment. Before moving day, she wants to decide where to place her furniture in her new living room. When she visited the apartment, she drew this rough sketch of the room's layout and recorded some measurements.



She has also measured her large furniture, which she wants placed by the movers. These measurements are shown in the table below.

Furniture	Dimensions (width by length)
couch	40 in. by 90 in.
loveseat	40 in. by 66 in.
wall unit	20 in. by 60 in.

? How can you use a **scale diagram** of this room, on an 8.5 in. by 11 in. sheet of paper, to determine where to place these pieces of furniture?

scale diagram

A drawing in which measurements are proportionally reduced or enlarged from actual measurements; a scale diagram is similar to the original.

scale

The ratio of a measurement on a diagram to the corresponding distance measured on the shape or object represented by the diagram.

A. Determine a **scale** you can use to create a scale diagram of the living room on an 8.5 in. by 11 in. sheet of paper.

B. Use your scale to determine what the lengths of walls and openings in your scale diagram should be.

C. Create your scale diagram of the living room.

D. Use your scale to determine the dimensions of each piece of furniture that needs to be placed.

E. Select a strategy to determine a good location for each piece of furniture. Add the three pieces of furniture to your scale diagram.

Reflecting

F. Compare your diagram with your classmates' diagrams. How are they the same, and how are they different?

G. Maxine used a **scale factor** of $\frac{1}{16}$ to create her diagram. Explain the advantages of using this scale factor.

H. Does it make sense that the scale factor you used for your diagram had to be less than 1? Explain.

scale factor

A number created from the ratio of any two corresponding measurements of two similar shapes or objects, written as a fraction, a decimal, or a percent.

Communication ***Tip***

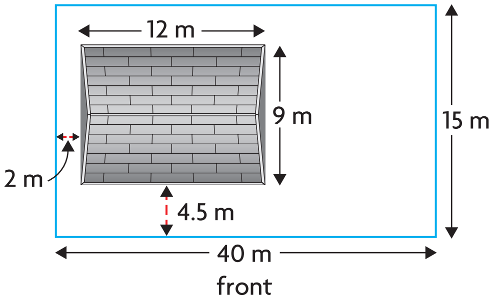
A scale is a ratio or rate, so it always includes units. A scale factor is a number without units.

APPLY the Math

EXAMPLE 1 Drawing a 2-D scale diagram that requires a reduction

A builder plans to construct a house on a rectangular lot, as shown in this sketch.

Draw a scale diagram of the lot and house using a scale of 1 m : 500 m.



Eric's Solution

Lot dimensions:

$$\text{Lot length} = 40\text{ m}\left(\frac{1}{500}\right)$$

$$\text{Lot length} = 0.08\text{ m}$$

$$\text{Lot width} = 15\text{ m}\left(\frac{1}{500}\right)$$

$$\text{Lot width} = 0.03\text{ m}$$

House dimensions:

$$\text{House length} = 12\text{ m}\left(\frac{1}{500}\right)$$

$$\text{House length} = 0.024\text{ m}$$

$$\text{House width} = 9\text{ m}\left(\frac{1}{500}\right)$$

$$\text{House width} = 0.018\text{ m}$$

Since the scale is 1 m : 500 m, which is less than 1, I knew that my scale diagram would be a reduction of the actual house and lot. The scale factor is

$$\frac{1\ \cancel{\text{m}}}{500\ \cancel{\text{m}}} = \frac{1}{500}$$

I multiplied all of the measurements by the scale factor.

Insets from left and front of lot:

$$\text{Front inset} = 4.5\text{ m}\left(\frac{1}{500}\right)$$

$$\text{Front inset} = 0.009\text{ m}$$

$$\text{Left inset} = 2\text{ m}\left(\frac{1}{500}\right)$$

$$\text{Left inset} = 0.004\text{ m}$$

Lot dimensions:

$$\text{Lot length} = 0.08\ \cancel{m}\left(\frac{100\text{ cm}}{1\ \cancel{m}}\right)$$

$$\text{Lot length} = 8.0\text{ cm}$$

$$\text{Lot width} = 0.03\ \cancel{m}\left(\frac{100\text{ cm}}{1\ \cancel{m}}\right)$$

$$\text{Lot width} = 3.0\text{ cm}$$

House dimensions:

$$\text{House length} = 0.024\ \cancel{m}\left(\frac{100\text{ cm}}{1\ \cancel{m}}\right)$$

$$\text{House length} = 2.4\text{ cm}$$

$$\text{House width} = 0.018\ \cancel{m}\left(\frac{100\text{ cm}}{1\ \cancel{m}}\right)$$

$$\text{House width} = 1.8\text{ cm}$$

All the measurements I calculated for my scale diagram are in metres, but my ruler is in centimetres. To draw the diagram accurately, I converted the measurements to centimetres.

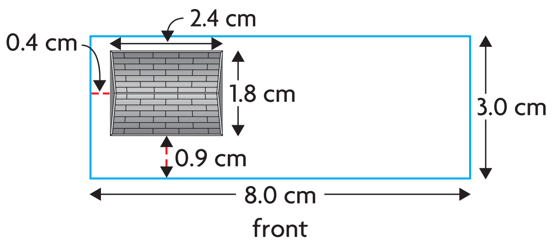
Insets from left and front of lot:

$$\text{Front inset} = 0.009\ \cancel{m}\left(\frac{100\text{ cm}}{1\ \cancel{m}}\right)$$

$$\text{Front inset} = 0.9\text{ cm}$$

$$\text{Left inset} = 0.004\ \cancel{m}\left(\frac{100\text{ cm}}{1\ \cancel{m}}\right)$$

$$\text{Left inset} = 0.4\text{ cm}$$



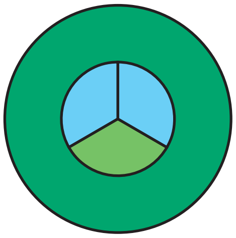
I used the measurements I calculated to draw the lot and then the house. I noticed that I could draw the rectangle for the house only one way, since the house had to be completely inside the lot.

Your Turn

Joe decided to draw a scale diagram of the lot and house using a scale factor of 0.01. Explain how his diagram would differ from Eric's diagram.

EXAMPLE 2 Drawing a 2-D scale diagram that requires an enlargement

Jess designed the logo shown for an environment club. She wants to enlarge the logo so that it can be applied to the front of a baseball cap. The hat company has suggested a scale factor of $\frac{5}{3}$. Draw a scale diagram of the logo as it will appear on the baseball cap.



Jess's Solution

Diameter of outer circle = 3 cm
Diameter of inner circle = 1.5 cm
Length of each line segment = 0.75 cm
Measure of all sector angles = 120°

I measured the diameters of the outer and inner circles on my logo design, and I recorded the measurements. I thought that the three line segments were radii of the inner circle, but I measured them just to be safe. I also measured the angle of each sector.

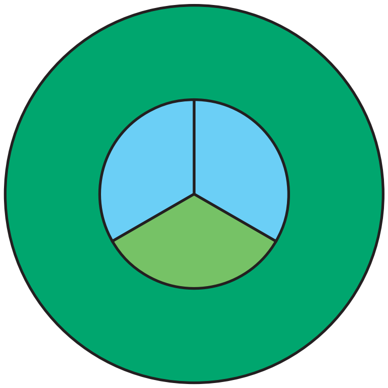
$$\text{New outer diameter} = 3 \text{ cm}\left(\frac{5}{3}\right) \text{ or } 5 \text{ cm}$$

$$\text{New inner diameter} = 1.5 \text{ cm}\left(\frac{5}{3}\right) \text{ or } 2.5 \text{ cm}$$

$$\text{New line segments} = 0.75 \text{ cm}\left(\frac{5}{3}\right) \text{ or } 1.25 \text{ cm}$$

Measure of all new sector angles = 120°

I calculated the new measurements for the enlarged logo by multiplying each linear measurement by the scale factor of $\frac{5}{3}$. Since the original logo and the enlarged logo are similar, I knew the measure of all the sector angles would be 120°.



I drew the larger circle by setting my compass radius to 5 cm. Then I drew the smaller circle by using the same centre and setting my compass radius to 2.5 cm. I drew the vertical radius of the inner circle and used a protractor to measure angles of 120° from this radius to draw the other two radii. Finally, I coloured in the enlarged logo using the same colours I used for my logo design.

Your Turn

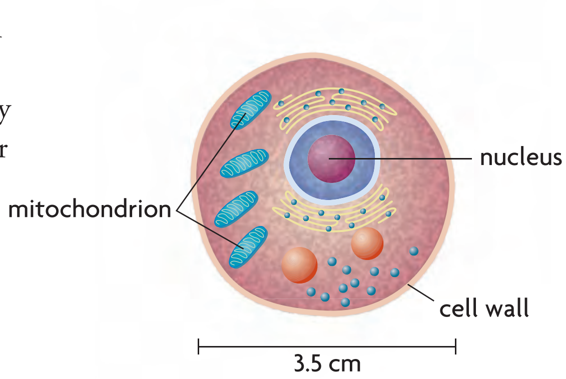
Jess initially thought of using a scale factor of 400%.

a) Draw the logo using this scale factor.

b) Why do you think she decided to use the recommended scale factor?

EXAMPLE 3 Determining scale factor

The diameter of the animal cell that is represented by this scale diagram is actually 0.25 mm. What scale factor was used to draw this scale diagram?



Hannah's Solution

Communication *Tip*

The variable *k* is often used to represent an unknown scale factor.

Let k be the scale factor of the diagram.

$$k = \frac{\text{Diagram measurement}}{\text{Actual measurement}}$$

$$k = \frac{35 \cancel{\text{mm}}}{0.25 \cancel{\text{mm}}}$$

The diameter of the cell in the diagram is 3.5 cm, and the actual diameter is 0.25 mm. I expressed both measurements in millimetres and then wrote a ratio.

$$k = \frac{35}{0.25}$$

$$k = \frac{3500}{25}$$

To eliminate the decimal, I multiplied both terms in the ratio by 100 and then simplified.

$$k = \frac{140}{1}$$

The scale factor used for the scale diagram is 140.

Your Turn

Explain why it makes sense that the scale factor used for this scale diagram is greater than 1.

In Summary

Key Ideas

- Scale diagrams can be used to represent 2-D shapes. To create a scale diagram, you must determine an appropriate scale to use. This depends on the dimensions of the original shape and the size of diagram that is required.
- The scale factor represents the ratio of a distance measurement of a shape to the corresponding distance measurement of a similar shape, where both measurements are expressed using the same units.

Need to Know

- You can multiply any linear dimension of a shape by the scale factor to calculate the corresponding dimension of a similar shape.
- When determining the scale factor, k, used for a scale diagram, the measurement from the original shape is placed in the denominator:

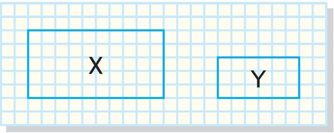
$$k = \frac{\text{Diagram measurement}}{\text{Actual measurement}}$$

- When a scale factor is between 0 and 1, the new shape is a reduction of the original shape.
- When a scale factor is greater than 1, the new shape is an enlargement of the original shape.

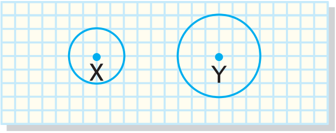
CHECK Your Understanding

1. Determine the scale factor that was used to transform diagram X into diagram Y. Express your scale factor as a fraction and as a percent.

a)



b)



2. Determine if the original will be larger or smaller than the scale diagram after the given scale factor is applied.

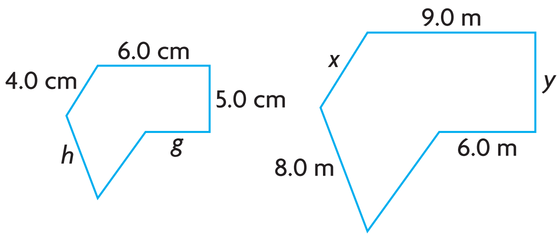
a) scale factor: 112% **b)** scale factor: 0.75 **c)** scale factor: $\frac{4}{9}$

3. On a plan, an actual length of 6 ft is represented by 5 in.

a) Determine the scale of the plan.

b) Determine the scale factor used to make the plan.

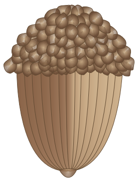
4. The following two polygons are similar. Determine the lengths of sides g, h, x, and y to the nearest tenth of a unit.



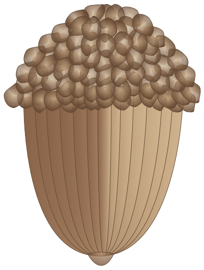
PRACTISING

5. The Garry oak is a tree that is found on Vancouver Island. The original acorn for these scale diagrams was 1.9 cm long. Determine the scale factor that was used for each diagram.

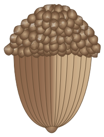
a)



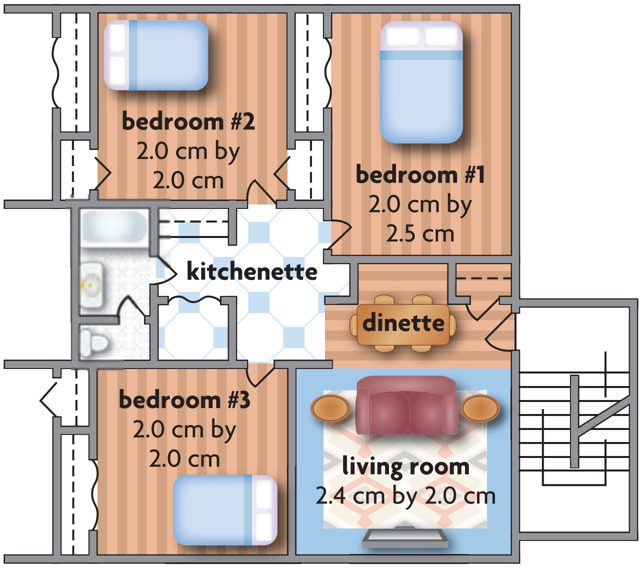
b)



c)

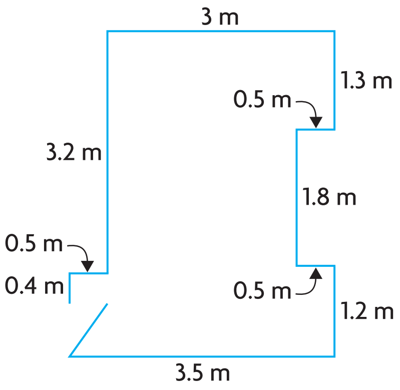


6. The floor plan of an apartment is shown to the right, drawn using a scale factor of 0.01.
   a) What are the actual dimensions of each bedroom?
   b) What are the actual dimensions of the living room?
   c) Which room has the greatest area?



7. Yani wants to make a scale diagram of the floor plan of his school. He wants his diagram to fit on an 8.5 in. by 11 in. sheet of paper. The school is 650 ft long and 300 ft wide at its widest point.
   a) What would be a reasonable scale for Yani to use so that his diagram will fit on the sheet of paper?
   b) Assume that the school's floor plan is a rectangle. Draw a scale diagram using the scale you determined in part a).

8. Ken has made a sketch of the floor plan of his bedroom. Draw an accurate scale diagram of his bedroom on 1 cm grid paper, using a scale factor of $\frac{1}{50}$.



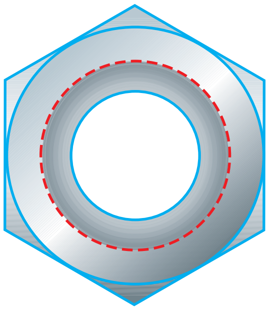
9. A computer chip on a circuit board has a rectangular shape, with a width of 6 mm and a length of 9 mm. Plans for the circuit board must be drawn using a scale factor of 15. Draw a scale diagram of the computer chip as it would appear on the plans.

10. This top view of a hex-nut must be enlarged by a scale factor of 250% for a display at a trade show.

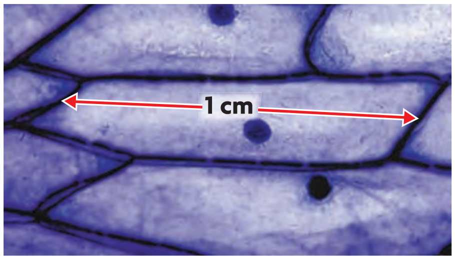
a) Measure the necessary distances on the diagram.

b) Determine what the corresponding distances on the enlarged diagram should be.

c) Draw the scale diagram of the hex-nut.



11. Sara has a microscope with a lens that magnifies by a factor of 40. She was able to capture the image of a slide containing onion cells, as shown. In the image, the cell was about 1 cm long. How long is the actual onion cell, to nearest hundredth of a millimetre?



12. a) Using the map scale, estimate the distance from

i) Yellowknife to Fort Norman

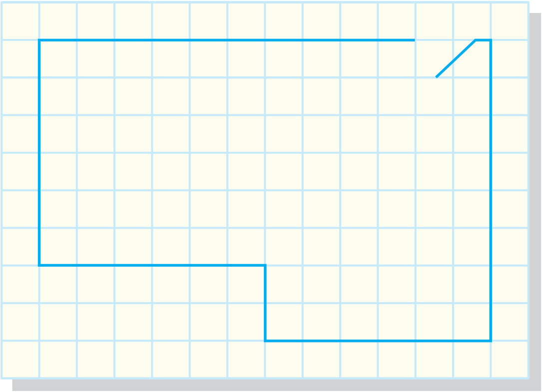
ii) Fort Providence to Fort Norman

b) Of the three locations on the map, which two are closest to each other?

13. This scale diagram, drawn on 0.5 cm grid paper, shows the floor plan of a greenhouse, drawn using a scale factor of 1 : 75.

a) Determine the perimeter of the greenhouse.

b) Determine the area of the floor of the greenhouse.

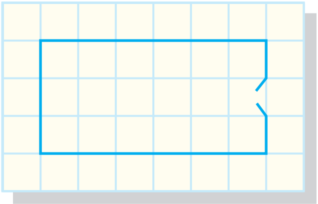


14. Determine the scale factor used in each situation.

- **a)** The actual diameter of a penny is 19 mm. In a scale diagram, the diameter of a penny is 5.7 cm.
- **b)** The actual width of a door is 30 in. In a scale diagram, the width of the door is $1\frac{1}{2}$ in.
- **c)** The diagonal of an actual stamp is 2.5 cm long. In a scale diagram, the diagonal is 1.0 m long.
- **d)** The height of an actual communications tower is 55 ft. In a scale diagram, the height of the tower is 6 in.

15. A billboard measures 4.5 m by 3.5 m. Draw a scale diagram of the billboard that fits in a space measuring 20 cm by 15 cm.

16. The floor plan for a garden shed is shown below. The area of the actual floor is 72 m^2.



- **a)** Determine the actual area that each square on the floor plan represents.
- **b)** Determine the actual distance that 5 mm on the floor plan represents.
- **c)** Determine the scale of the plan.
- **d)** Determine the scale factor that was used to draw the floor plan.

17. The viewing areas of most LCD televisions are similar rectangles. Regardless of the size of a television, the length : width ratio is often 16 : 9. Rahj has built-in bookshelves that are 4 ft wide. There is a vertical distance of 26 in. between each shelf. Show that a 42 in. LCD television will fit on one of these shelves.

18. **a)** Use geometric shapes to create a logo that will fit in a space measuring 12 cm by 12 cm.

- **b)** Draw a scale diagram of your logo using a scale factor of 25%.
- **c)** Draw a scale diagram of your logo using a scale factor of 1.5.

Closing

19. When drawing a scale diagram on a sheet of paper, how do you decide what scale factor to use? What do you need to consider?

Extending

20. Sanjay has 34 in. of red oak moulding that is 1 in. wide. He would like to use this moulding to frame a photograph, as shown. The photograph measures 12 in. by 8 in., so the frame would require more moulding than he has.

- **a)** By what scale factor should he reduce the photograph so that he can use the wood he has to make the frame?
- **b)** What are the dimensions of the reduced photograph?

8.4

Scale Factors and Areas of 2-D Shapes

GOAL

Solve area problems that involve similar 2-D shapes.

INVESTIGATE the Math

Quilting is as old as ancient Egypt, if not older. For most of its history, however, quilting was used to make clothing. Pieced quilts, made by sewing pieces of fabric into blocks and then sewing together the blocks, are a more recent development.

Norma is making a quilt by sewing together congruent pieces of cloth. To create a larger quilt, she sews together more congruent shapes. She makes sure that the larger quilt is similar in shape to the original quilt.

? How does the area of the larger quilt relate to the area of the original quilt?

A. Suppose that Norma uses square pieces of fabric. Use square pattern blocks to represent these pieces of fabric.

Measure the dimensions of one square, and determine its area. Create a table like the one below, and record the dimensions and area.

Length (in.)	Width (in.)	Area (in.^{2})

B. Starting with one square, add enough squares to create a larger square that has double the original dimensions. Record its dimensions and area in your table.

C. Add more squares to create a larger square that has three times the original dimensions. Record the dimensions and area of the larger square in your table.

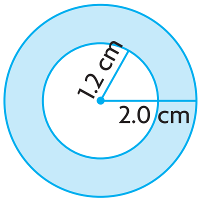
D. Predict the area of a square that has four times the original dimensions. Check your prediction, and record the dimensions and area of this square in your table.

E. As the square grows larger, how does its area relate to the scale factor k and the area of the original square?

YOU WILL NEED

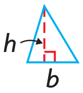
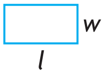
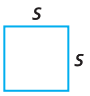
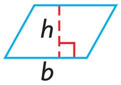
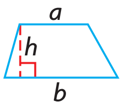
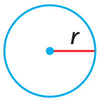
- calculator
- ruler
- pattern blocks

EXPLORE…



- Determine the area of the shaded region. If both radii are doubled, does the area also double?

Area Formulas

Shape	Formula
triangle 	$A = \frac{1}{2}bh$
rectangle 	$A = lw$
square 	$A = s^2$
parallelogram 	$A = bh$
trapezoid 	$A = \frac{1}{2}h(a + b)$
circle 	$A = \pi r^2$

F. Suppose that Norma uses congruent triangular pieces of fabric. Repeat parts A to E using triangular pattern blocks and a table like the one below.

Base (in.)	Height (in.)	Area (in.^{2})

G. Suppose that Norma uses congruent rectangular pieces of fabric. Repeat parts A to E using pieces of U.S. letter or U.S. legal paper and a table like the one shown in part F.

H. Make a **conjecture** about the relationship among the area of a shape, the scale factor, and the area of a larger similar shape.

Reflecting

I. Do you think your conjecture will hold when you decrease the dimensions of a shape by a specific scale factor? Explain.

J. Do you think your conjecture will hold for other similar shapes, such as parallelograms, trapezoids, or circles? Explain.

K. Do you think your conjecture will hold for any pair of similar 2-D shapes? Explain.

APPLY the Math

EXAMPLE 1 Reasoning about scale factor and area

Jasmine is making a kite from a 2 : 25 scale diagram. The area of the scale diagram is 20 cm^2. How much fabric will she need for her kite?

Jasmine's Solution: Reasoning about scale as an enlargement

$$\text{Scale factor} = \frac{25}{2} \text{ or } 12.5$$

As the scale factor is greater than 1, the kite is an enlargement of the scale diagram.

$$k = 12.5$$

k represents the scale factor for the enlargement.

$$\text{Area of kite} = k^2(\text{Area of scale diagram})$$
$$\text{Area of kite} = (12.5)^2(20 \text{ cm}^2)$$
$$\text{Area of kite} = 3125 \text{ cm}^2$$

I know that the scale diagram and the actual kite are similar shapes. This means that the area of the actual kite can be determined by multiplying the area of the scale diagram by the square of the scale factor.

I will need at least 3125 cm^2 of fabric for my kite.

This amount of fabric will cover the frame exactly. I'll need more than this amount, since I'll have to sew the fabric to the frame.

Hank's Solution: Reasoning about scale as a reduction

Let k represent the scale factor.

$$k = \frac{2}{25}$$

The scale diagram is a reduction of the kite since the scale factor is less than 1.

$$\text{Area of scale diagram} = k^2(\text{Area of kite})$$
$$\frac{\text{Area of scale diagram}}{\text{Area of kite}} = k^2$$

Since the scale diagram and the actual kite are similar shapes, the area of the scale diagram equals the product of the square of the scale factor k and the area of the actual kite.

Let x represent the area of the kite.

$$\frac{20\text{ cm}^2}{x} = \left(\frac{2}{25}\right)^2$$
$$\frac{20\text{ cm}^2}{x} = \frac{4}{625}$$
$$12\,500\text{ cm}^2 = 4x$$
$$3125\text{ cm}^2 = x$$

I substituted the information I knew into the equation.

Jasmine will need at least 3125 cm^2 of fabric for her kite.

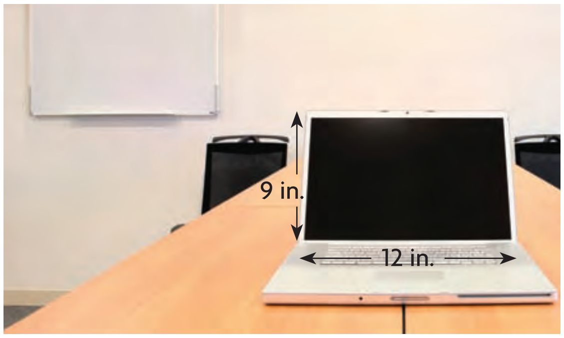
This amount of fabric will cover the frame exactly. She will need a little more than this amount, so that she can sew the fabric to the frame.

Your Turn

If the scale diagram for the kite had been drawn using a scale ratio of 1 : 20, and the area of the scale diagram had been 30 cm^2, how much fabric would Jasmine have needed for her kite?

EXAMPLE 2 Reasoning about scale factor and area to determine dimensions

Jim's laptop has a monitor with the dimensions 9 in. by 12 in. The image on his laptop is projected onto the screen of a whiteboard. According to the documentation for the whiteboard, its screen area is 2836.6875 in.^2.



a) The image on the whiteboard is similar to the image on the laptop. Determine the scale factor used to project the images on the laptop to the whiteboard.

b) Determine the dimensions of the whiteboard.

Rani's Solution

a) Area of monitor $= lw$

Area of monitor $= (9\text{ in.})(12\text{ in.})$

Area of monitor $= 108\text{ in.}^2$

Let k represent the scale factor.

Area of whiteboard $= k^2$(Area of monitor)

$$2836.6875\text{ in.}^2 = k^2(108\text{ in.}^2)$$

$$\frac{2836.6875\ \cancel{\text{in.}^2}}{108\ \cancel{\text{in.}^2}} = k^2$$

$$26.265... = k^2$$

$$\sqrt{26.265...} = k$$

$$5.125 = k$$

The laptop's monitor is a rectangle, so I determined its area by multiplying its length, *l*, and width, *w*.

The image on the laptop and the image on the whiteboard are similar rectangles. This means that the area of the image on the whiteboard is equal to the square of the scale factor times the area of the image on the laptop.

Since the image on the whiteboard is larger than the original, I know that $k > 1$. A scale factor of 5.125 makes sense.

The dimensions of the image on the whiteboard are an enlargement of the dimensions of the image on Jim's laptop by a factor of 5.125.

b) Let x represent the length of the whiteboard.

$x = (12\text{ in.})(5.125)$

$x = 61.5\text{ in.}$

Let y represent the width of the whiteboard.

$y = (9\text{ in.})(5.125)$

$y = 46.125\text{ in.}$

The whiteboard is about 61 in. long by 46 in. wide.

To determine the dimensions of the whiteboard, I multiplied the length and width of the laptop's monitor by the scale factor.

Your Turn

A circular icon on Jim's laptop has a diameter of 2 cm. Calculate the area of this icon on the whiteboard.

In Summary

Key Idea

- If two 2-D shapes are similar and their dimensions are related by a scale factor k, then the relationship between the area of the similar shape and the area of the original shape can be expressed as:

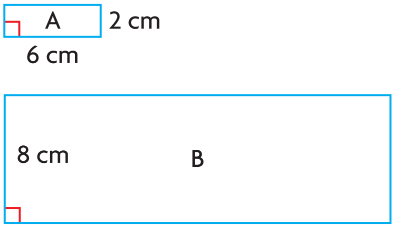
$$\text{Area of similar 2-D shape} = k^2(\text{Area of original shape})$$

Need to Know

- If the area of a similar 2-D shape and the area of the original shape are known, then the scale factor, k, can be determined using the formula

$$k^2 = \frac{\text{Area of similar 2-D shape}}{\text{Area of original shape}}$$

CHECK Your Understanding



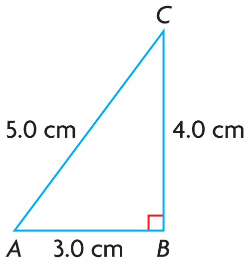
1. Two similar rectangles, A and B, are shown to the right.

a) Determine the scale factor that produced the enlargement from rectangle A to rectangle B.

b) Determine the areas of rectangle A and rectangle B.

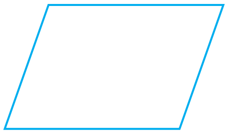
c) How many rectangles congruent to rectangle A would fit in rectangle B?

2. The table below gives data for enlargements and reductions of the triangle shown to the right. Complete the table.



Length of Base (cm)	Height of Triangle (cm)	Scale Factor	Area (cm^2)	$\frac{\text{Area of scaled triangle}}{\text{Area of original triangle}}$
3.0	4.0	1	6.0	1
		3		
1.5				
			600.0	
		25%		

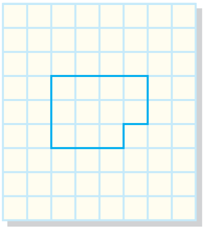
PRACTISING



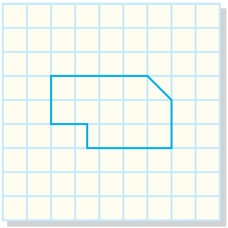
3. The parallelogram shown to the right has an area of 42 cm^2. It is going to be enlarged by a scale factor of 5. Determine the area of the enlarged parallelogram.

4. Determine the area of each figure after it is enlarged by a scale factor of 2.

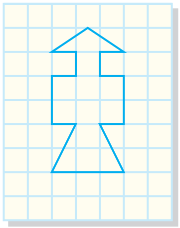
a)



b)

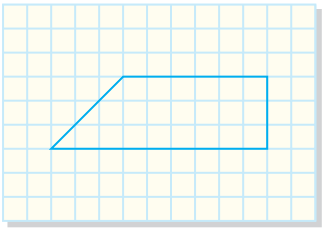


c)

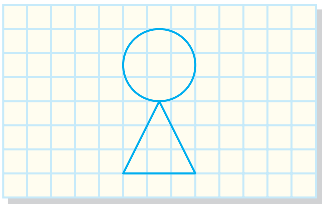


5. Determine the area of each figure, to the nearest tenth of a square unit, after it is reduced by a scale factor of $\frac{1}{3}$.

a)



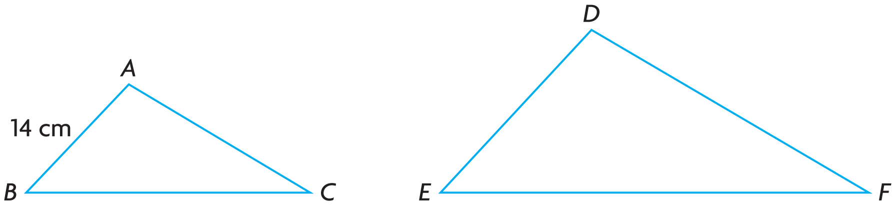
b)



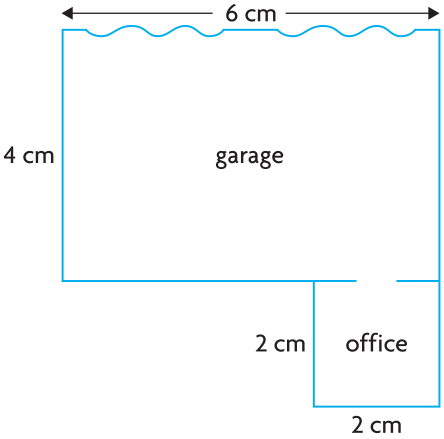
6. Tammy downloaded a photograph, which measured 4 in. by 6 in., from her camera to her laptop. Then Tammy used a software program to enlarge the dimensions of the photograph by 150% so that it would fit in a frame she already had.
   a) What are the inside dimensions of the frame she already had?
   b) By what percent was the area of the photograph increased in the enlarging process?
   c) Explain how you could determine the area of the enlarged photograph using two different strategies.

7. Stop signs on city, town, and rural roads are regular octagons. Describe how you would create a similar stop sign that is quadruple the area of a typical stop sign for increased visibility on a two-lane highway.

8. $\triangle ABC$ and $\triangle DEF$ are similar triangles. The sum of the lengths of AB and DE is 35 cm. The area of $\triangle DEF$ is 144 cm^2.



   a) Determine the scale factor that relates $\triangle ABC$ to $\triangle DEF$.
   b) Determine the area of $\triangle ABC$.



9. The sketch to the left of a service garage and an attached office was drawn using a scale ratio of 1 : 500. On this diagram, the area of the garage is 24 cm^2 and the area of the office is 4 cm^2. Determine the area of the actual garage and the actual office in square metres.

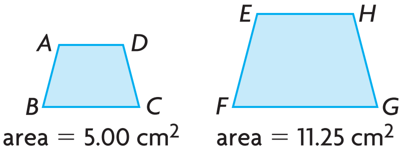
10. A rectangular display, with the dimensions 2 m by 3 m, is located in the lobby of city hall to show the citizens the layout for the new People's Park. The display was created using a scale ratio of 1 : 120.
   a) The parks department estimates that the city spends \$0.75/m^2 to maintain a park from spring through fall. Estimate the cost to maintain People's Park.
   b) A rectangular model, with the same dimensions, was used to represent Meadow Park. The scale ratio used was 1 : 250. Estimate the cost to maintain Meadow Park.

11. A gymnasium wall is 20 ft high and 120 ft long. Peggy has been asked to paint a mural on the wall. The mural must be $\frac{1}{4}$ the area of the wall and the mural and wall must be similar. The mural must also be centred on the wall. Draw a scale diagram that shows the dimensions of the wall, the dimensions of the mural, and where the mural should be placed.

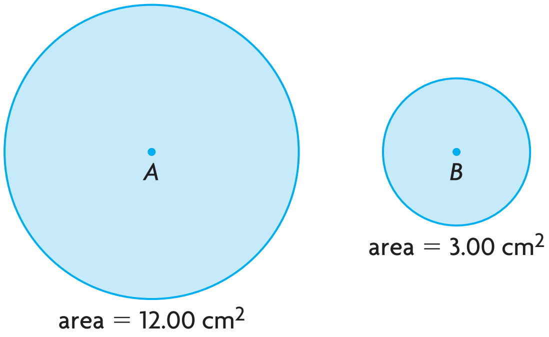
12. The scale factor for two similar rectangles is 1 : 2. The sum of their areas is 40 cm^2. Determine the area of each rectangle.

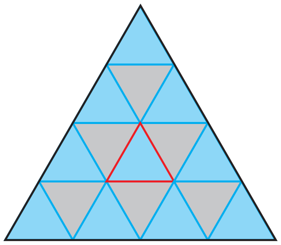
13. Determine the scale factor that relates each pair of similar shapes.

a)



b)



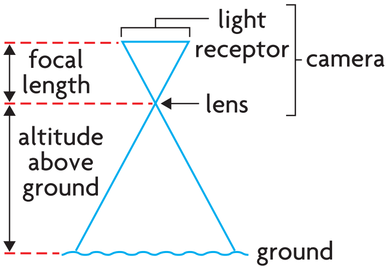


14. In the diagram to the right, the large triangle, outlined in black, is an enlargement of the small triangle, outlined in red. The small triangle is congruent to the other small triangles, which are equilateral and have side lengths of 1 unit.

a) Determine the value of the scale factor, *k*.

b) Explain how *k* relates the perimeters and areas of the large and small triangles.

15. Aerial photographs are often used to show parcels of land that are for sale. The camera used to take the photograph below had a focal length of 0.152 m. The altitude of the airplane was 7600 m when the photograph was taken. The ratio of the camera's focal length to the airplane's altitude is the scale factor for the photograph.



a) Determine the scale of the aerial photograph.

b) Determine the area of the parcel of land shown in the photograph in hectares. The conversion rate is 1 ha per 10 000 m^2.

c) Determine the value of the parcel of land, if it sells for $375/ha.

16. You would like to renovate the kitchen in your home, and you need to create floor plans of the renovations for the contractor.
   - **a)** Measure and record the dimensions of your kitchen. Include doors and measurements of any counters, appliances, or furniture that take up floor space.
   - **b)** On one piece of paper, draw two scale diagrams: the first diagram showing the existing kitchen and the second diagram showing how you would like the kitchen to appear after the renovation.
   - **c)** Compare the area of the open floor space in the two versions of the kitchen. Which version is more spacious?

Closing

17. Explain the difference between the following processes:

   A. Reduce a 2-D shape by a scale factor of $\frac{1}{2}$.

   B. Divide the area of the same 2-D shape by 2.

   Use examples to support your explanation.

Extending

18. A polygon has its dimensions increased by 180% to create a similar polygon. The dimensions of the new polygon are then reduced by 50% to create a third polygon. What percent of the area of the original polygon is the area of the third polygon?

19. A company that manufactures cardboard boxes currently makes boxes with dimensions of 12 in. by 16 in. by 12 in. The company plans to make a new, larger box by increasing the current dimensions by 150%. Cardboard costs $0.05/sq ft. How much more will it cost to make the larger box?

8.5 Similar Objects: Scale Models and Scale Diagrams

GOAL

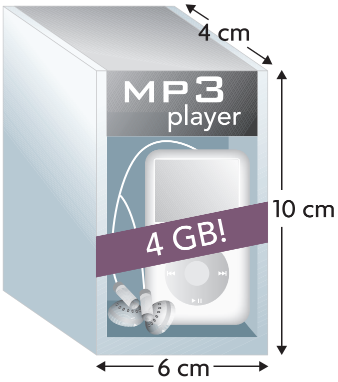
Understand and use scale models and scale diagrams that involve 3-D objects.

YOU WILL NEED

- calculator
- ruler
- centimetre grid paper
- scissors
- tape

INVESTIGATE the Math

Sameer is an engineer for an electronics company. His company currently sells a popular mini-MP3 player. It comes packaged in a box with the dimensions shown.



His team of engineers has designed a larger version of the mini-MP3 player, which has improved features and greater storage capacity. A new box must be created for this new MP3 player.

? How can you create a new box that is similar to the original box?

A. Make a net for the original box.

B. Choose a reasonable scale factor that you can apply to the net you made in part A in order to create a larger box.

C. Make the new net for the larger box.

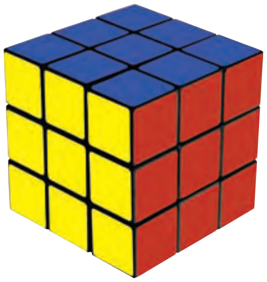
D. Use your net to construct a model of the larger box.

E. How are the dimensions of the boxes related?

F. Are your boxes **similar objects**? Explain.

EXPLORE…

- How do the dimensions of each small cube relate to the overall dimensions of the Rubik's Cube®?



similar objects

Two or more 3-D objects that have proportional dimensions.

Reflecting

G Are the lengths of the diagonals of each pair of corresponding faces in both boxes proportional?

H. Compare the new box you created to the new boxes created by your classmates. Are all of these boxes similar? Explain.

I. Juan claims that other kinds of objects, such as triangular prisms, cylinders, pyramids, cones, and spheres, can be similar, provided that they have the same shape. Do you agree or disagree? Explain.

J. Is it possible for two irregular 3-D objects to be similar? Explain.

APPLY the Math

EXAMPLE 1 Determining if two objects are similar

Sandeep is a chef. In his restaurant, he uses frying pans of various sizes. Are his frying pans similar?

Sandeep's Solution

$$\frac{\text{Bottom diameter of large pan}}{\text{Bottom diameter of small pan}} = \frac{30\text{ cm}}{20\text{ cm}} \text{ or } \frac{3}{2}$$

They look similar. To check, I measured corresponding parts of the pans and compared my measurements.

$$\frac{\text{Depth of large pan}}{\text{Depth of small pan}} = \frac{6\text{ cm}}{4\text{ cm}} \text{ or } \frac{3}{2}$$

$$\frac{\text{Handle length of large pan}}{\text{Handle length of small pan}} = \frac{24\text{ cm}}{16\text{ cm}} \text{ or } \frac{3}{2}$$

The two pans are similar objects. The large pan is an enlarged version of the small pan, by a factor of 1.5.

The corresponding measurements of the pans are proportional.

Your Turn

The top diameter of the large pan is 33 cm. Determine the top diameter of the small pan.

EXAMPLE 2 Determining actual dimensions from a scale model

Esmerelda bought this toy tractor to give to her younger brother for his birthday. The dimensions of the toy are given in the diagram to the right. The scale ratio on the package is 1:16. She knows that her brother will want to know the size of the real tractor. How can she determine the dimensions of the real tractor?

Esmerelda's Solution

The scale model is similar to the real tractor.

I know that all scale models are similar to the real objects. Their measurements are proportional to the corresponding measurements of the real objects.

The scale factor is $\frac{1}{16}$.

I need to multiply each of the dimensions of the model by 16.

Since the model is a reduction of the real tractor, the real tractor is an enlargement of the model by a scale factor of 16.

Actual height $= 16(12.7 \text{ cm})$
Actual height $= 203.2 \text{ cm}$

Actual width $= 16(9.5 \text{ cm})$
Actual width $= 152.0 \text{ cm}$

Actual length $= 16(19.1 \text{ cm})$
Actual length $= 305.6 \text{ cm}$

Actual height $= 2.032 \text{ m}$
Actual width $= 1.520 \text{ m}$
Actual length $= 3.056 \text{ m}$

I converted the measurements from centimetres to metres by multiplying by $\frac{1 \text{ m}}{100 \text{ cm}}$. This made the numbers more meaningful for my brother.

The actual height of the real tractor is about 2.0 m, the actual width is about 1.5 m, and the actual length is about 3.1 m.

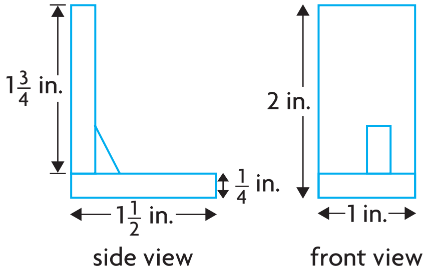
Your Turn

The diameter of the rear tires on the model is 6.0 cm. What is the diameter of the rear tires on the real tractor?

EXAMPLE 3 Enlarging from a scale diagram to determine actual dimensions

Nadia has found plans for a bookend in a woodworking magazine. The plans include a scale diagram, with a scale ratio of 1:5.

Determine the dimensions (length, width, and height) of the actual bookend.



Nadia's Solution

3-D scale diagrams are similar to the real objects.

I know that all 3-D scale diagrams are similar to the real objects. Their measurements are proportional to the corresponding measurements of the real objects.

The scale factor is

$$\text{Diagram : Actual} = 1:5 \text{ or } \frac{\text{Diagram}}{\text{Actual}} = \frac{1}{5}$$

To determine the measurements of the actual bookend from the scale diagram, I need to determine the reciprocal of the scale factor given. Since $\frac{5}{1}$ is greater than 1, the actual bookend is an enlargement of the scale diagram.

$$\text{Base length} = \left(1\frac{1}{2}\text{ in.}\right)5$$

$$\text{Base length} = \frac{15}{2} \text{ or } 7\frac{1}{2}\text{ in.}$$

I multiplied each dimension in the diagram by 5 to determine the actual dimensions of the bookend.

$$\text{Base thickness} = \left(\frac{1}{4}\text{ in.}\right)5$$

$$\text{Base thickness} = \frac{5}{4} \text{ or } 1\frac{1}{4}\text{ in.}$$

$$\text{Base width} = (1\text{ in.})5$$
$$\text{Base width} = 5\text{ in.}$$

$$\text{Overall height} = \left(1\frac{3}{4}\text{ in.} + \frac{1}{4}\text{ in.}\right)5$$

$$\text{Overall height} = (2\text{ in.})5 \text{ or } 10\text{ in.}$$

The dimensions of the actual bookend are $7\frac{1}{2}$ in. by 5 in. by 10 in., with a base thickness of $1\frac{1}{4}$ in.

Your Turn

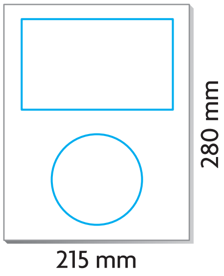
What would the dimensions of the bookend have been if the scale ratio on the plans had been 2:9?

EXAMPLE 4 Drawing a scale diagram of a 3-D object

Céline is an engineer. She is working on a city project, replacing old storm-sewer pipes with new concrete pipes. Each pipe has an inner diameter of 1.50 m, a wall thickness of 0.18 m, and a length of 2.5 m. How can she create a scale drawing of one of these pipes?

Céline's Solution

Drawing the entire pipe, as I see it, involves perspective. This will distort the actual measurements. Drawing a side view and a front view of the pipe will enable me to use proportional measurements.



I decided to split my piece of paper in half. I will use the top half to draw a scale diagram of the side view of the pipe, and the bottom half to draw a scale diagram of the front view.

Paper width:Actual length of pipe $= 215$ mm:2500 mm
The ideal scale ratio for the width of the paper is about 1:12.

I compared the width of the paper to the length of the pipe, in millimetres.

Paper length:2(Actual diameter of pipe) $= 280$ mm:3500 mm
The ideal scale ratio for the length of the paper is also about 1:12.

To check that the diagram would fit on the length of the paper, I compared the length of the paper to the diameter of two pipes, with allowance for a space between them. I estimated that this would be about 3.5 m before scaling.

Using a scale factor of $\frac{1}{20}$:

I decided to round down to $\frac{1}{20}$. This scale factor is less than $\frac{1}{12}$, and 20 is a number that is easy to divide into the actual measurements, resulting in numbers I can draw line segments for accurately.

Scale diagram pipe length $= \frac{2.5 \text{ m}}{20}$ or 0.125 m

Scale diagram inner diameter $= \frac{1.5 \text{ m}}{20}$ or 0.075 m

Scale diagram wall thickness $= \frac{0.18 \text{ m}}{20}$ or 0.009 m

I divided each of the actual measurements by 20 to determine the corresponding measurements on the scale diagram.

Scale diagram pipe length $= 125$ mm
Scale diagram inner diameter $= 75$ mm
Scale diagram wall thickness $= 9$ mm

Since the smallest unit on my ruler is mm, I multiplied each measure by $\frac{1000 \text{ mm}}{1 \text{ m}}$ to convert the measurements to millimetres.

Side view:

Length $= 125$ mm

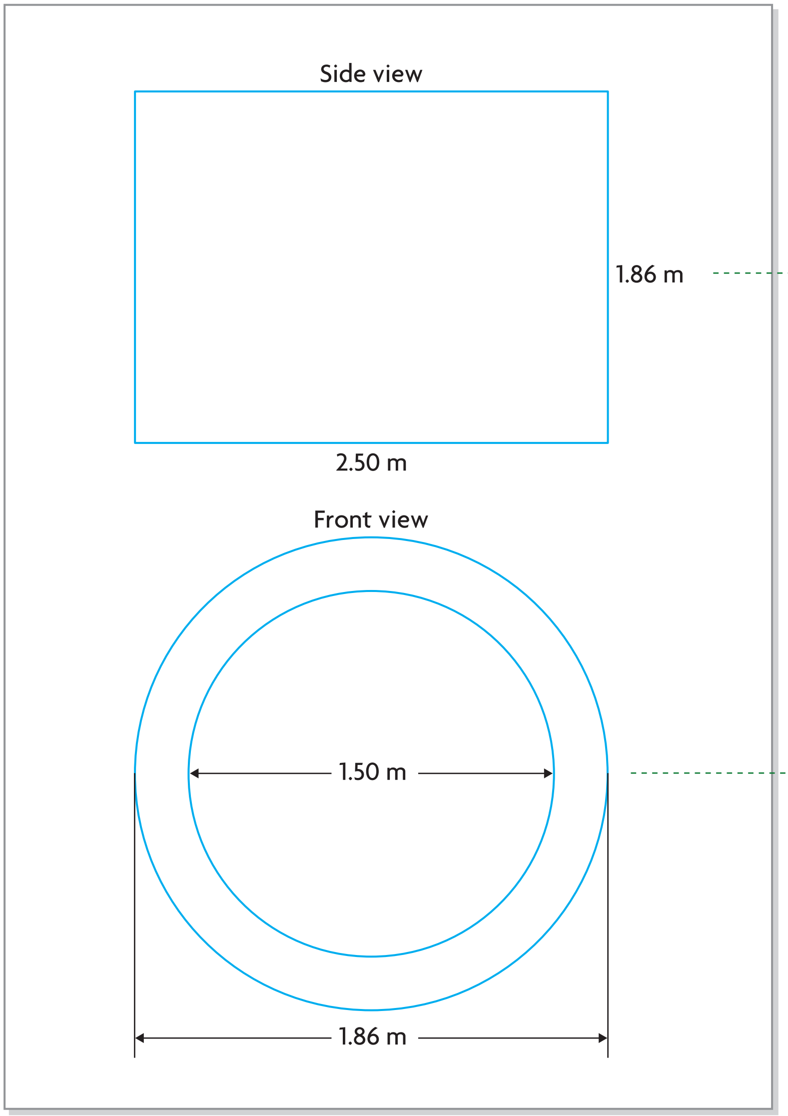
Width $= 75$ mm $+ 9$ mm $+ 9$ mm or 93 mm

The width of the pipe is the sum of the inner diameter and twice the thickness of its wall.

Front view:

Inner diameter $= 75$ mm

Outer diameter $= 93$ mm



Shown at 50% of actual size

The side view of the pipe will look like a rectangle. I added the inner diameter to the wall thickness at the top and bottom of the pipe to determine the width of the pipe.

The front view looks like two circles. The outer diameter corresponds to the width of the rectangle.

Your Turn

Draw a scale drawing of the pipe using a scale factor of $\frac{1}{15}$.

In Summary

Key Ideas

- Two 3-D objects that are similar have dimensions that are proportional.
- The scale factor is the ratio of a linear measurement of an object to the corresponding linear measurement in a similar object, where both measurements are expressed using the same units.
- To create a scale model or diagram, determine an appropriate scale to use based on the dimensions of the original shape and the size of the model or diagram that is required.

Need to Know

- You can multiply any linear measurement of an object by the scale factor to calculate the corresponding measurement of the similar object.
- You can determine the scale factor k, used to create a scale model of an object by using any corresponding linear measurements of the object and the scale model:

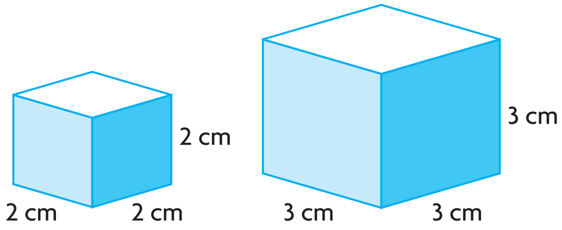
$$k = \frac{\text{Linear measurement of scale model}}{\text{Corresponding linear measurement of object}}$$

- When a scale factor is between 0 and 1, the new object is a reduction of the original object.
- When a scale factor is greater than 1, the new object is an enlargement of the original object.

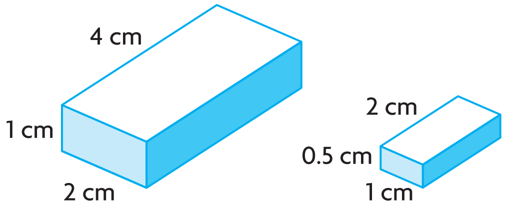
CHECK Your Understanding

1. For each of the following, determine whether the two objects are similar and justify your decision.

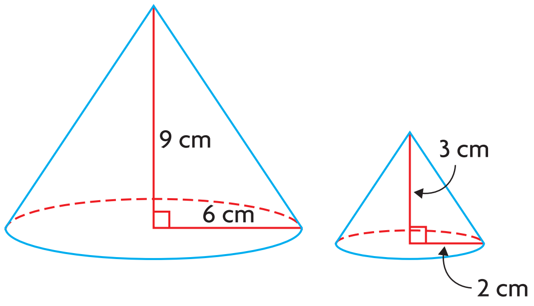
a)



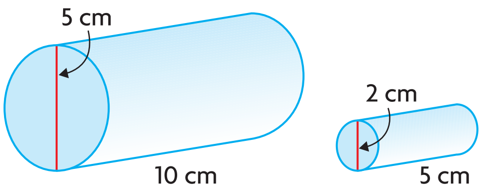
b)



c)



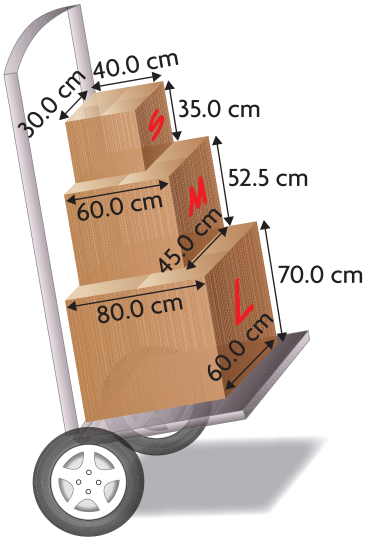
d)



2. The National Basketball Association (NBA) uses a basketball with a diameter of 25 cm. The Women's National Basketball Association (WNBA) uses a basketball with a diameter of 22 cm.
   a) Are these balls similar? Explain.
   b) Determine the scale factor that relates
      i) the NBA ball to the WNBA ball
      ii) the WNBA ball to the NBA ball

3. One of the most famous ships in Canadian sailing history is the *Bluenose*. Launched in 1921 from Nova Scotia as a fishing vessel, the *Bluenose* operated in the rough waters off the coast of Newfoundland. The *Bluenose* became very famous for its speed, winning all the great classic sailing races on the American east coast. Mark has a 1:100 scale model of this ship. The model has a length of 52 cm, a beam (width) of 8.5 cm, and a height of 43 cm. Calculate the length, beam, and height of the actual ship.

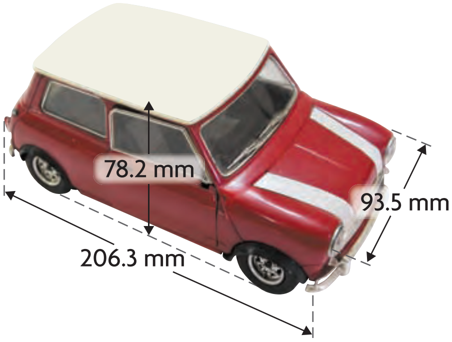
PRACTISING



4. Toni works for a moving company. The company sells three different-sized boxes, as shown.
   a) Are the boxes similar? Explain.
   b) The letters on the boxes (S, M, L for small, medium, large) increase in proportion to the size of the box. The red M on the medium box is 24 cm tall. Determine the heights of the S and the L.

5. Last summer, Ed visited the Royal Tyrrell Museum in Drumheller, Alberta, to see the fossil and dinosaur exhibits. While he was there, he purchased a 1:40 scale model of the *Albertosaurus libratus*, which was native to the area over 70 million years ago. The length of the model is 21.5 cm, and the height is 9.5 cm. Determine the length and height of this species of dinosaur.

6. A 1:18 scale model of a car has the dimensions shown. Determine the dimensions of the actual car.



7. The bald eagle is Canada's largest bird of prey. It has a body length of about 90 cm and, while perched, a height of about 75 cm. Hank is a woodcarver who wants to create a carving, to scale, of a bald eagle while perched. He has a block of wood that is 150 cm long, 150 cm wide, and 200 cm high.

a) Suggest a scale factor that Hank could use for his carving.

b) If he uses the scale factor you suggested in part a), determine the height and length of the eagle he will create.

8. A carving of Tecumseh, the Shawnee leader of a confederacy that fought in the war of 1812, is located in the Wood Carving Museum in Windsor, Ontario. The carving is $6\frac{1}{2}$ ft tall by $2\frac{1}{2}$ ft wide. The museum wants to sell replica models that are 26 in. tall in the gift shop.

a) What scale factor must be used to produce these models?

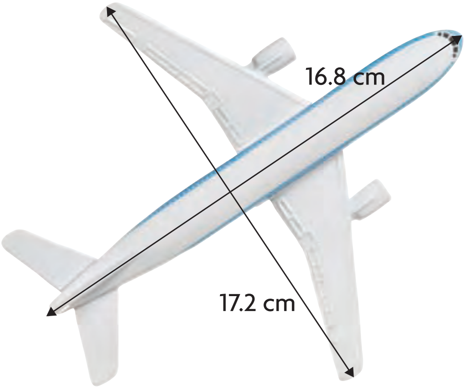
b) Determine the width of these models.

9. Umiaks are boats that are used in the Arctic for transportation and for traditional whale hunting. The frame of an umiak is built from spruce wood. Traditionally, the outer cover was made from animal skins, such as walrus and bearded-seal skins, but today it can be made from ballistic nylon. A typical umiak is 32 ft long, with a beam (width) of 48 in. Determine these dimensions on a scale model built using a scale ratio of 1:24.

10. Some model train enthusiasts enjoy building villages on their layouts. Two popular scale ratios for model trains are HO (1:87) and N (1:160). Nick has found a building that he would like to add to his N-scale layout, but its dimensions are for an HO-scale layout. The dimensions are 6 in. long by $8\frac{1}{2}$ in. tall by 4 in. wide.

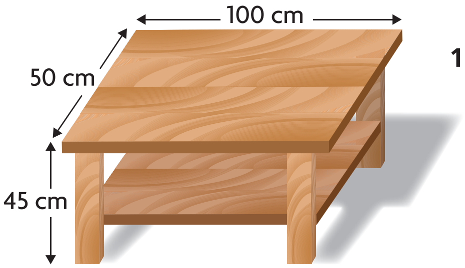
a) Estimate what the dimensions of the building would be in N scale.
b) Determine the conversion ratio for HO:N.
c) Determine the dimensions of the building in N scale, to the nearest eighth of an inch.

11. The measurements of a scale model of a passenger jet are shown. The model was made using a scale factor of $\frac{1}{200}$. The floor of Hangar 77 at the Calgary International Airport measures 46.6 m long by 71.9 m wide. How many of these passenger jets could fit in this hangar?



12. Take a photograph of a structure or building, as well as a referent—an object with a known height or length. For example, you could include a metre stick or another student in your photograph.

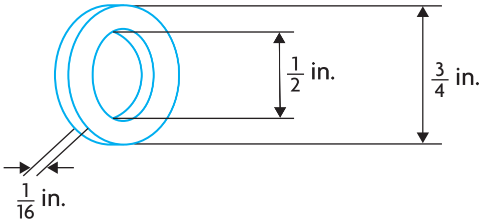
a) Estimate the measurements of the structure or building using only your photograph.
b) Describe how you used a scale factor to determine the measurements.



13. Draw a scale diagram that shows the top, front, and side views of this coffee table. Assume a uniform thickness of 5 cm for all the pieces of wood. Each leg is inset 10 cm from the edge of the tabletop, and the bottom shelf is 10 cm above the ground.

Use a scale factor of $\frac{1}{10}$.

14. The specifications for a steel washer are shown.



Draw a scale diagram of the top, side, and front views of the washer, using a scale ratio of 4:1.

15. This chest freezer has the dimensions indicated. Draw a scale diagram of the top, side, and front views of the freezer, so that all three views fit on a single page of standard paper.

16. Choose an object in your classroom, and measure its dimensions. Draw a scale diagram of the object that shows top, side, and front views.

17. Suppose that you increase the dimensions of a box, which is the shape of a rectangular right prism, by 150%.

a) Do you think the area of the base of the box will also increase by this scale factor? Justify your decision.

b) Will the volume of the box also increase by this scale factor? Justify your decision.

Closing

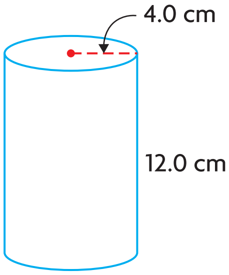
18. How is the process for solving problems that involve similar objects the same as the process for solving problems that involve similar shapes? How is the process different?

Extending

19. A juice company plans to enlarge this can by a scale factor of 1.5.

a) The new can will be made from the same metal, in the same thickness, as the smaller can. By what factor will the cost of the metal increase?

b) The cost of the metal that is needed to make the larger can is \$0.045. Determine the cost of the metal that is needed to make the smaller can.



20. The surface area of a right cone is 100 cm^2. Its dimensions are reduced by 50% to produce a similar cone. Determine the surface area of the similar cone.

8.6 Scale Factors and 3-D Objects

YOU WILL NEED

- calculator
- ruler
- linking cubes
- pattern blocks
- rectangular blocks

EXPLORE…

- Suppose that you are trying to decide whether you should order an 8 in. pizza or a 16 in. pizza. How much more pizza will you have if you order the 16 in. pizza instead of the 8 in. pizza? What assumptions are you making?

GOAL

Solve problems that involve scale factor, surface area, and volume.

INVESTIGATE the Math

Rostrum blocks are used as props in drama productions. They are three-dimensional objects of various sizes, such as cubes, right rectangular prisms, right triangular prisms, and right cylinders. Many of these objects have a horizontal top surface to stand on, sit on, lean on, or rest other props on.

Kayley, a set carpenter, has been asked to create sets of similar rostrum blocks. She needs to know the surface areas of the blocks, so she can determine how much material she will require to build them. She also needs to know their volumes, so she can predict how much space they will take up in the storage area between shows.

? What is the relationship between the scale factor and the surface areas of two similar objects? What is the relationship between the scale factor and the volumes of two similar objects?

A. Suppose that Kayley wants to make a set of similar cubes. Use linking cubes to act as models.

Measure the dimensions of one linking cube, and determine its surface area and volume. Create a table like the one below, and record your findings.

Length (cm)	Width (cm)	Height (cm)	Surface Area (cm^2)	Volume (cm^3)

B. Starting with a single cube, add enough cubes to create a larger cube with dimensions that are double the dimensions of the single cube. Record the dimensions, surface area, and volume of the larger cube in your table.

C. Add more cubes to create a cube with dimensions that are three times greater than the dimensions of a single cube. Record the dimensions, surface area, and volume of the larger cube in your table.

D. Predict the surface area and volume of a cube with dimensions that are four times greater than those of a single cube. Check your predictions, and then record the dimensions, surface area, and volume in your table.

E. As the cube grows larger, how does its surface area relate to the scale factor k and the surface area of the original cube? How does its volume relate to the scale factor k and the volume of the original cube?

F. Suppose that Kayley wants to create a set of right triangular prisms. Repeat parts A to E using triangular pattern blocks and a table like the one below.

Side Length of Base (in.)	Height of Triangular Base (in.)	Height of Prism (in.)	Surface Area (in.^{2})	Volume (in.^{3})

G. Suppose that Kayley wants to create a set of right prisms with rectangular bases. Repeat parts A to E using rectangular blocks and a table like the one shown in part F.

H. **i)** Make a **conjecture** about the relationship among the surface area of the original object, the scale factor, and the surface area of a larger similar object.

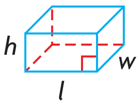
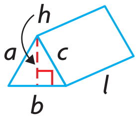
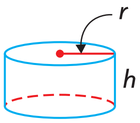
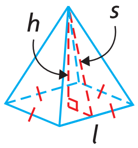
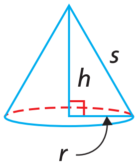
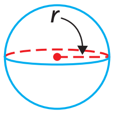
ii) Make a conjecture about the relationship between the volume of the original object, the scale factor, and the volume of a larger similar object.

Reflecting

I. Do you think your conjectures will hold when you decrease the dimensions of an object by a specific scale factor? Explain.

J. Do you think your conjectures will hold for other similar objects, such as pyramids, cones, cylinders, or spheres? Explain.

K. Do you think your conjectures will hold for any pair of similar 3-D objects? Explain.

Formulas	
Object	**Surface Area and Volume**
rectangular prism 	$SA = 2(lw + lh + wh)$ $V = lwh$
right triangular prism 	$SA = bh + l(a + b + c)$ $V = \frac{1}{2}bhl$
right cylinder 	$SA = 2\pi r^2 + 2\pi rh$ $V = \pi r^2 h$
right pyramid 	$SA = l^2 + 2ls$ $V = \frac{1}{3}l^2 h$
right cone 	$SA = \pi r^2 + \pi rs$ $V = \frac{1}{3}\pi r^2 h$
sphere 	$SA = 4\pi r^2$ $V = \frac{4}{3}\pi r^3$

APPLY the Math

EXAMPLE 1 Reasoning about relationships among scale factor, surface area, and volume

Prove the scaling conjectures for the surface area and volume of a rectangular right prism with dimensions $l \times w \times h$.

Connor's Solution: Proving the conjecture for surface area

The surface area of the original rectangular right prism, SA_{original}, can be expressed as

I decided to prove the conjecture for surface area.

$$SA_{\text{original}} = 2(lw + lh + wh)$$

Let k be the scale factor.
The surface area of a new scaled rectangular right prism, SA_{new}, can be expressed as

The rectangular right prism will be scaled by a factor of k. To ensure that the scaled prism is similar to the original, each of the dimensions of the original prism must be multiplied by k.

$l_{\text{new}} = k \cdot l_{\text{original}}$
$w_{\text{new}} = k \cdot w_{\text{original}}$
$h_{\text{new}} = k \cdot h_{\text{original}}$

$$SA_{\text{new}} = 2[(kl)(kw) + (kl)(kh) + (kw)(kh)]$$

$$SA_{\text{new}} = 2[k^2(lw) + k^2(lh) + k^2(wh)]$$

$$SA_{\text{new}} = 2k^2[(lw) + (lh) + (wh)]$$

$$SA_{\text{new}} = k^2(SA_{\text{original}})$$

k^2 is a common factor.

The expression inside the brackets is the same as the expression for the surface area of the original prism.

The conjecture is valid for the surface area of a rectangular right prism.

Isabelle's Solution: Proving the conjecture for volume

The volume of the original rectangular right prism, V_{original} can be expressed as

I decided to prove the conjecture for volume.

$$V_{\text{original}} = lwh$$

Let k be the scale factor.
The volume of a new scaled rectangular right prism, V_{new}, can be expressed as

The rectangular right prism will be scaled by a factor of k, so each of the dimensions will be multiplied by k.

$$V_{\text{new}} = (kl)(kw)(kh)$$

$$V_{\text{new}} = k^3lwh$$

$$V_{\text{new}} = k^3(lwh)$$
$$V_{\text{new}} = k^3(V_{\text{original}})$$

The expression inside the brackets is the same as the expression for the volume of the original prism.

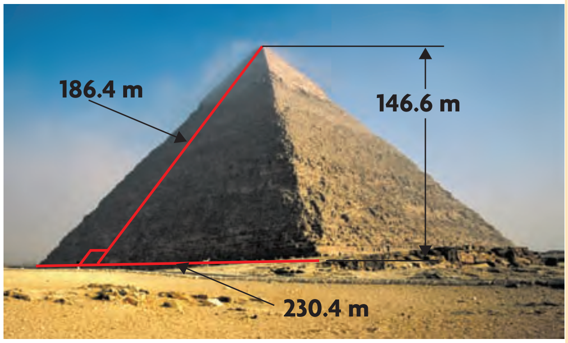
The conjecture is valid for the volume of a rectangular right prism.

Your Turn

Prove the scaling conjectures for the surface area and volume of a sphere.

EXAMPLE 2 Solving a surface area problem

The Great Pyramid of Giza in Egypt was built on a square base, with the dimensions shown.



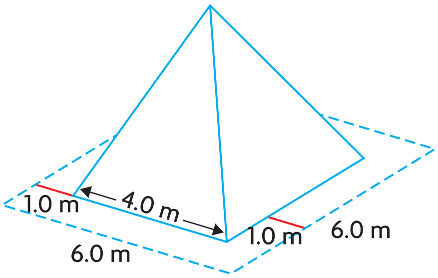
An artist who works with plate glass wants to build a replica of the pyramid for an installation at an art gallery. The artist is restricted by the floor dimensions, which are 6.0 m by 6.0 m, and the ceiling height of 3.5 m. As well, the glass sculpture must have room for a 1.0 m walkway around its base.

a) What scale factor might the artist use to build the sculpture?

b) How much glass will the artist need to build the sculpture?

Twila's Solution

a)



I drew a diagram to show the situation. The artist's sculpture will be a pyramid that is similar to the actual pyramid in Giza. Since 1.0 m of clearance is needed on each side of the sculpture's base, the longest possible side length is 4.0 m.

$$k = \frac{\text{Side length for base of original pyramid}}{\text{Side length for base of sculpture}}$$
$$k = \frac{230.4 \cancel{m}}{4.0 \cancel{m}}$$
$$k = 57.6$$
$$k \doteq 60$$

Since I knew the measures of the side lengths of both bases, I chose to determine a scale factor, *k*, that could be used to create the sculpture. Then I rounded up my answer to a convenient value. If I had rounded down, the sculpture would not have fit, as $\frac{1}{60}$ is less than $\frac{1}{57.6}$.

I'll test a scale factor of $\frac{1}{60}$ to determine the height of the sculpture from the height of the original pyramid.

The height of the sculpture must be less than 3.5 m.

Height of sculpture = (Scale factor)(Height of original pyramid)

$$\text{Height of sculpture} = \left(\frac{1}{60}\right)(146.6\text{ m})$$

I checked that the height of the sculpture would fit in the display area.

Height of sculpture = 2.443... m

2.4 m is less than 3.5 m, so the sculpture will fit.

The artist could use a scale factor of $\frac{1}{60}$.

b) $$\frac{SA_{\text{sculpture}}}{SA_{\text{original}}} = k^2$$

I knew the dimensions of the original pyramid. I can determine the surface area of the sculpture, $SA_{\text{sculpture}}$, by calculating the surface area of the original, SA_{original}.

$$SA_{\text{sculpture}} = k^2(SA_{\text{original}})$$

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2(SA_{\text{original}})$$

Since the base is a square, each of the triangular faces is congruent.

The expression in square brackets is the formula for the surface area of a square-based pyramid.

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2[\text{Area of base} + 4(\text{Area of each triangular face})]$$

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2\left[b^2 + 4\left(\frac{bs}{2}\right)\right]$$

In the original pyramid, *b* is length of one of the sides of the square base, which is 230.4 m. In the original pyramid, *s* is the slant height (altitude) of each triangular face, which is 186.4 m.

$$SA_{\text{sculpture}} = \left(\frac{1}{60}\right)^2\left[(230.4\text{ m})^2 + 4\left(\frac{(230.4\text{ m})(186.4\text{ m})}{2}\right)\right]$$

$$SA_{\text{sculpture}} = 38.604...\text{ m}^2$$

The artist will need about 38.6 m^2 of glass to build the sculpture in the given space.

Your Turn

The gift shop at the art gallery would like to sell miniature replicas of the artist's sculpture. A scale ratio of 1 : 50 will be used to make the replicas.

a) Determine the dimensions of the replicas.

b) Determine the amount of glass that will be needed to make each replica.

EXAMPLE 3 Solving a capacity problem

The smaller tank in the photograph has a capacity of 1400 m^3, and the larger tank has a capacity of 4725 m^3.

a) During the refining process, both tanks are filled with oil from a pumping station at the same rate. How many times longer will it take to fill the larger tank than it will take to fill the smaller tank?

b) How many times greater is the radius of the larger tank than the radius of the smaller tank?

Spherical tanks are often used to store oil and gas at refineries, since this shape is the most economical to build.

Esther's Solution

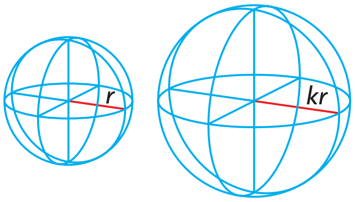
a)
$$\frac{V_{\text{large}}}{V_{\text{small}}} = \frac{4725\ \cancel{m^3}}{1400\ \cancel{m^3}}$$
$$\frac{V_{\text{large}}}{V_{\text{small}}} = 3.375$$

It will take a little more than three times longer to fill the larger tank than it will take to fill the smaller tank.

Since both tanks are being filled at the same rate, the factor that relates the capacities of the tanks will tell me the relationship between the times needed to fill the tanks. I decided to use V_{large} to represent the capacity (interior volume) of the large tank and V_{small} to represent the capacity of the small tank.

The larger tank holds 3.375 times more oil than the smaller tank.

b)



$$V_{\text{large}} = k^3 \cdot V_{\text{small}}$$
$$\frac{V_{\text{large}}}{V_{\text{small}}} = k^3$$
$$3.375 = k^3$$
$$\sqrt[3]{3.375} = k$$
$$1.5 = k$$

The inner radius of the larger tank is 1.5 times greater than the inner radius of the smaller tank.

The only dimension that varies in a sphere is the radius. Therefore, any two spheres are similar objects, with radii related by a scale factor of k.

The capacities of the tanks are related by a factor of 3.375. Therefore, the scale factor that relates the radii of the tanks is the cube root of this number.

Your Turn

The larger tank is going to be reduced by a scale factor of 0.6 to build another small tank. Determine the capacity of the new small tank.

In Summary

Key Idea

- If two 3-D objects are similar and their dimensions are related by the scale factor k, then
  - Surface area of similar object $= k^2$(surface area of original object)
  - Volume of similar object $= k^3$(volume of original object)

Need to Know

- If you know the dimensions of a scale diagram or model of a 3-D object, as well as the scale factor used to enlarge/reduce from the diagram or model, you can determine the surface area and volume of the enlarged/reduced object, without determining its dimensions.

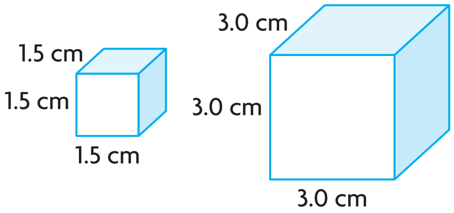
CHECK Your Understanding

1. Each pair of objects is similar.

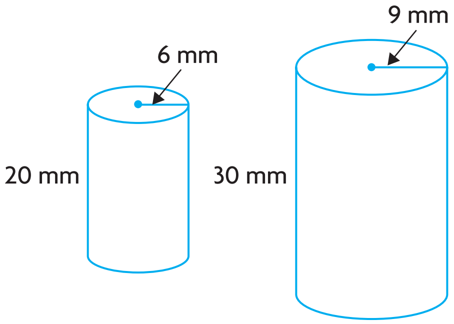
i) By what factor is the surface area of the larger object greater than the surface area of the smaller object?

ii) By what factor is the volume of the larger object greater than the volume of the smaller object?

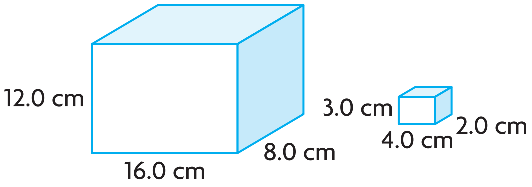
a)



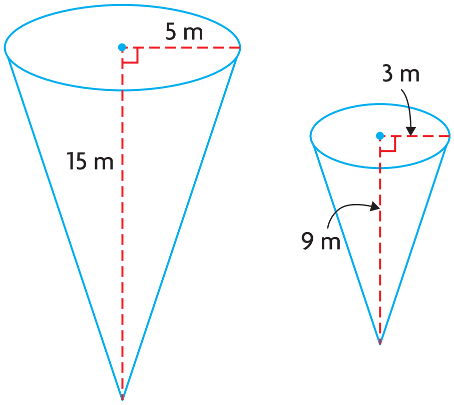
b)

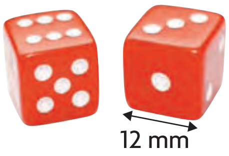


c)



d)





actual dice

2. A stage director needs a pair of large dice for a scene with children playing a board game. He estimates that the measure of each edge of each enlarged die must be 600 mm.

a) What scale factor must he apply to create the enlarged dice?

b) How many times greater will the surface area of each larger die be?

c) How many times greater will the volume of each larger die be?

3. A model of a ship is built to a scale ratio of 1 : 30. The model is 16 cm tall, and the area of one sail is 8.5 cm^2. What are the corresponding measurements of the actual ship?

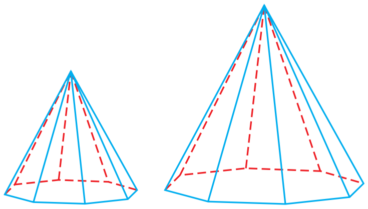
PRACTISING

4. An oil tank has a capacity of 32 m^3. A similar oil tank has dimensions that are larger by a scale factor of 3. What is the capacity of the larger tank?

5. A soft-cover book will be modified so that it has large print for people who are visually impaired. To maintain the same number of pages, both the print size and page dimensions will be tripled.
- **a)** The area of each page in the original book is 500 cm^2. Determine the area of each page in the large-print book.
- **b)** The same type of paper will be used for the pages in the large-print book. By what factor will the volume of the paper change? Justify your answer.

6. Brenda is a potter. She is creating two similar vases, with their dimensions related by a scale factor of $\frac{3}{4}$. The larger vase has a volume of 9420 cm^3. Determine the volume of the smaller vase.

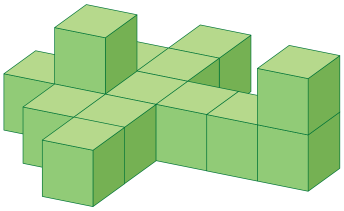
7. The dimensions of a right octagonal pyramid are enlarged by a scale factor of 1.5. Determine the value of each of the following ratios.
- **a)** $\dfrac{\text{Volume of large pyramid}}{\text{Volume of small pyramid}}$
- **b)** $\dfrac{\text{Surface area of large pyramid}}{\text{Surface area of small pyramid}}$
- **c)** $\dfrac{\text{Base perimeter of large pyramid}}{\text{Base perimeter of small pyramid}}$

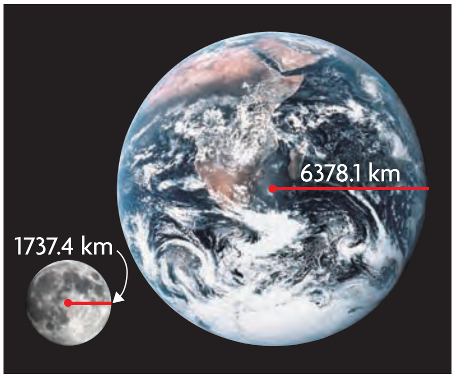


8. A jewellery box has a volume of 4500 cm^3. Its lid has a surface area of 375 cm^2. If each dimension of the jewellery box is tripled to create a prop for a theatre production, by what factors would the surface area of the lid and the volume of the box increase?

9. Celine's grandmother brought her a set of Russian dolls from St. Petersburg. The dolls stack inside each other and are similar to each other. The diameters of the two smallest dolls are 2.0 cm and 3.5 cm. The scale factor is the same from each doll to the next larger doll. Celine estimates the smallest doll has a volume of about 8 cm^3. Estimate the volume of the largest of the five dolls.

10. Mario made a scale model of an airplane using linking cubes.
- **a)** How many linking cubes would he need to make a model with dimensions five times as large?
- **b)** By what factor is the surface area of the new model greater than the surface area of the first model?





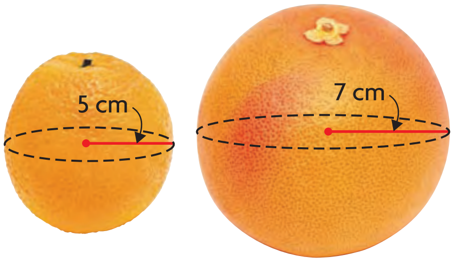
11. Adele wants to compare Earth and the Moon by creating spherical models. She has decided to represent Earth with a sphere that has a radius of 10.0 cm.

a) What is the radius of the sphere she should use to represent the Moon? Round your answer to the nearest tenth of a centimetre.

b) Determine the ratio that compares the circumference of the model of Earth to the circumference of the model of the Moon.

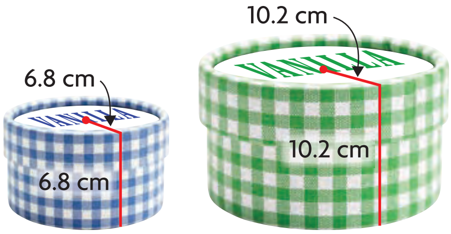
c) Determine the ratio that compares the surface area of the model of Earth to the surface area of the model of the Moon.

d) Determine the ratio that compares the volume of the model of Earth to the volume of the model of the Moon.



12. Markian likes both oranges and grapefruits. He wonders how much more fruit he gets in a grapefruit. Estimate how many times greater the volume of a grapefruit is, compared with the volume of an orange.

13. A baseball has a diameter of about 2.9 in. A softball has a diameter of about 3.8 in. By what percent is the amount of leather needed to cover the softball greater than the amount of leather needed to cover the baseball?



14. Josephine packages her ice cream in right cylindrical cardboard containers, in the two different sizes shown.

a) Determine the factor by which the height of the letters in "VANILLA" differs on the two containers.

b) Determine the factor by which the surface areas of the lids differ.

c) Determine the factor by which the capacities of the two containers differ.

d) How much ice cream does each container hold?

15. Suppose that your class is sending shoeboxes filled with school supplies to schools in need after a devastating earthquake. A cardboard manufacturer has donated two sizes of shoeboxes. The small shoebox is 18.0 cm long, 11.5 cm wide, and 9 cm high. The large shoebox is 36.0 cm long, 23 cm wide, and 18 cm high.

a) Matty claims that it will take about twice as much paper to wrap the large shoebox for shipping. Do you agree? Justify your decision.

b) Is the volume of the small shoebox half the volume of the large shoebox? Explain how you know.

16. Show that when the dimensions of each given object are enlarged/reduced by a scale factor of k, the surface area of the resulting similar object has changed by a factor of k^2 and its volume has changed by a factor of k^3.

a) right cylinder

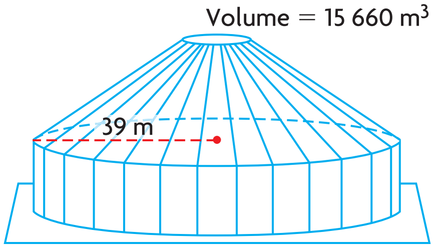
b) right cone

Closing

17. Two similar right rectangular prisms have the dimensions 3 m by 4 m by 2 m and 6 m by 8 m by 4 m. Explain how you can determine the number of smaller prisms that will fit in the larger prism.

Extending

18. A travelling circus holds performances under a large tent, as shown. A smaller version of this circus, which visits smaller communities, uses a similar tent with a volume of 580 m^3. How many times greater is the floor area of the larger tent, compared with the floor area of the smaller tent?



19. A manufacturer has created a spherical model of the Moon, using a scale ratio of 1 : 11 580 000. The model fits exactly into a cubic box with a volume of 27 000 cm^3.

a) Determine the surface area of the Moon.

b) Determine the volume of the Moon.

20. A bakery sells two sizes of birthday cakes. The small cake has a diameter of 10 in., and the large cake has a diameter of 12 in. The small cake sells for $14.00. What should the price of the large cake be? Justify your answer, and state any assumptions you are making.

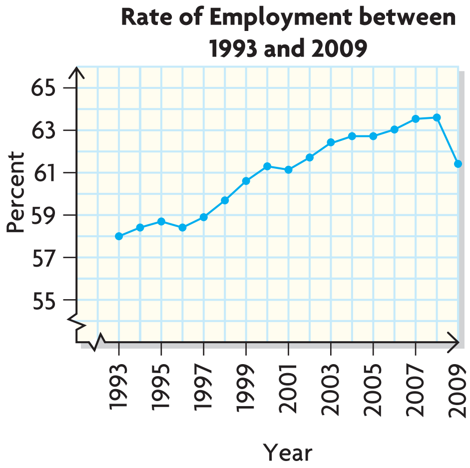
Math in Action

Are prices of TVs scaled appropriately?

Some products are available in different sizes. Sometimes these products are similar, like flat-screen TVs. TVs are measured by the diagonal of the viewing area. Some common sizes are 32 in., 40 in., and 52 in. The ratio of the length to the width of the viewing area is 16 : 9.

- Work with a partner or in a small group.
- Determine the dimensions of the viewing areas of the three sizes of TVs mentioned above.
- Research to find prices of these sizes of TVs from the same manufacturer.
- Determine if the prices are related by the scale factor.
- Do manufacturers consider scale factor when they set prices? Summarize the results of your findings and give possible reasons for what you found.

8 Chapter Self-Test



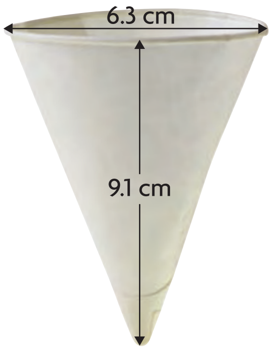
Source: Statistics Canada, Labour Force Survey

1. This graph shows the percent of Canadians who worked (the rate of employment) between 1993 and mid-2009.

- **a)** When was the rate of employment increasing? When was it decreasing?
- **b)** Identify when the employment rate did not change.
- **c)** When was the employment rate increasing the fastest? When was it increasing the slowest?
- **d)** Suggest some factors that may influence how this rate changes with time.

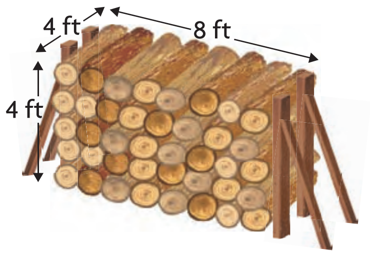
2. A model airplane has a scale ratio of 1 : 500. To paint the model, Michael used 5 mL of paint. Assuming the same kind of coverage for the paint, how many litres of paint would be needed to paint the actual airplane?

3. A company's logo has a rectangular shape, which measures 4 cm by 6 cm, on its letterhead. The company would like to advertise in the community and has decided to put an enlargement of the logo on the boards in the local hockey rink. The size of the logo is restricted to an area no larger than 1.5 m^2. Determine the greatest dimensions that the company could use.



4. To save money, a company that makes drinking cups for dentists has decided to reduce the size of its cups by a scale factor of $\frac{3}{4}$.

- **a)** Determine the radius, r, and the height, h, of the new cup.
- **b)** Determine the slant height, s, of the new cup.
- **c)** How much material does the company save for each cup it makes?
- **d)** When filled, approximately how much less water does the new cup hold, compared with the original cup?



5. A cord of wood is stacked in a pile with a volume of 128 ft^3. Three cords of wood will be stacked in a pile with dimensions that are similar to the dimensions of the first cord. Predict how much longer the second pile will be, compared with the original pile.

6. A pizza shop advertises cheese or pepperoni pizzas at the prices listed in the table.

- **a)** Is the scale factor used for price the same as the scale factor used for the diameter, area, or volume? Explain.
- **b)** Determine the "best buy"—the diameter that gives the most pizza for the least cost.

Diameter (in.)	Price ($)
10	6
12	8
14	10

***WHAT** DO You Think Now?* Revisit **What Do You Think?** on page 443. How have your answers and explanations changed?

8 Chapter Review

FREQUENTLY ASKED Questions

Study Aid

- See Lesson 8.3, Examples 1 to 3.
- See Lesson 8.5, Examples 1 to 3.
- Try Chapter Review Questions 5 to 7 and 12 to 14.

Q: What are scale diagrams and scale models, and what are they used for?

A: A scale diagram is a drawing that is a reduction or an enlargement of a 2-D shape or 3-D object. A scale model is a reduced or enlarged model of a 3-D object. A scale diagram or scale model is similar to the actual shape or object. To create a scale diagram or model, the same scale factor is applied to all the linear measurements.

For example, to create a model of a train using a scale ratio of 1 : 40, multiply every linear measurement of the train by the scale factor of $\frac{1}{40}$ or 0.025. Scale factors less than 1 (such as 1 : 40, $\frac{1}{40}$, 0.025, and 2.5%) indicate a reduction. Scale factors greater than 1 (such as 3 : 2, $\frac{3}{2}$, 1.5, and 150%) indicate an enlargement.

Scale diagrams and scale models allow you to visualize or handle shapes or objects that might otherwise be too large or too small to see and manipulate.

Some objects, such as buildings, may be represented using multiple 2-D scale diagrams. For example, plans for a building allow you to compare the sizes of rooms, decide where doors should be placed, and so on, before spending money on construction.

Study Aid

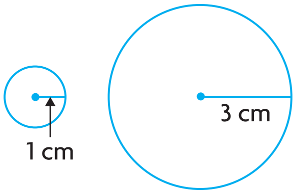
- See Lesson 8.4, Examples 1 and 2.
- Try Chapter Review Questions 8 to 10.

Q: How are the areas of two similar shapes related?

A: When two shapes are similar, their corresponding dimensions are proportional. The ratio of a pair of corresponding dimensions is a number called the scale factor. The scale factor is often represented by k. The areas of two similar shapes are related by the square of the scale factor.

Area of similar shape $= k^2$(Area of original shape)

For example, these two circles are similar.



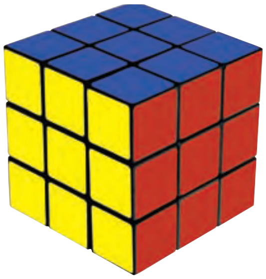
The scale factor that relates the dimensions of the small circle to those of the large circle is 3. So,

Area of large circle $= 3^2$(Area of small circle)

Area of large circle $= 9$(Area of small circle)

Study ***Aid***

- See Lesson 8.6, Examples 1 and 2.
- Try Chapter Review Questions 13 to 16.



Q: How are the surface area and volume of similar objects related?

A: Consider the Rubik's Cube shown. It consists of 27 individual cubes, each similar to the Rubik's Cube itself. The scale factor between the length of a side of an individual cube and the length of a side of the Rubik's Cube is $\frac{3}{1}$ or 3.

The Rubik's Cube has a surface area of (6)(9) or 54 square units. Each individual cube has a surface area of 6 square units. The ratio of the surface areas is $\frac{54}{6}$ or 9. This is the value of the scale factor squared.	The Rubik's Cube has a volume of (3)(3)(3) or 27 cubic units. Each individual cube has a volume of 1 cubic unit. The ratio of the volumes is $\frac{27}{1}$ or 27. This is the value of the scale factor cubed.
The surface area of the similar object, SA_{similar}, is related to the original object, SA_{original}, by the square of the scale factor. $SA_{\text{similar}} = k^2(SA_{\text{original}})$	The volume of the similar object, V_{similar}, is related to the original object, V_{original}, by the cube of the scale factor. $V_{\text{similar}} = k^3(V_{\text{original}})$

Study ***Aid***

- See Lesson 8.6, Example 3.
- Try Chapter Review Question 16.

Q: If you know the surface areas or volumes of two similar objects, how can you determine the scale factor that relates their dimensions?

A:

Area increases/decreases by the square of the scale factor, k, that relates the original object to the similar object. Therefore, the scale factor is the square root of the ratio of the surface areas.	Volume increases/decreases by the cube of the scale factor, k, that relates the original object to the similar object. Therefore, the scale factor is the cube root of the ratio of the volumes.
For example, in the Rubik's Cube, $k^2 = \frac{\text{Surface area of Rubik's Cube}}{\text{Surface area of small cube}}$ $k^2 = \frac{54}{6}$ $k = \sqrt{9}$ $k = 3$	For example, in the Rubik's Cube, $k^3 = \frac{\text{Volume of Rubik's Cube}}{\text{Volume of small cube}}$ $k^3 = \frac{27}{1}$ $k = \sqrt[3]{27}$ $k = 3$

PRACTISING

Lesson 8.1

1. An athlete runs the first lap of a race slightly faster than the second lap, and then runs the final lap the fastest. Draw a distance versus time graph that compares the athlete's average speed on each lap.

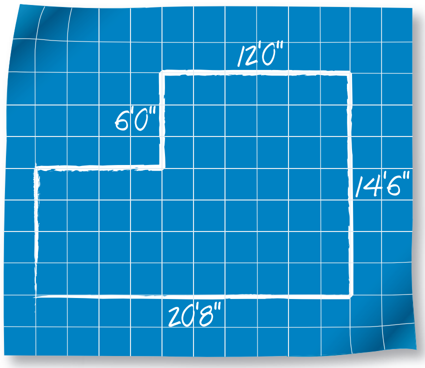
2. For each of the following, compare the two rates and determine the lower rate.

- **a)** frozen hams: \$2.58/kg or \$0.226/100 g
- **b)** cycling speeds: 35 km/h or 15 min to travel 4.5 mi
- **c)** fuel efficiency: 6.5 L/100 km or 38 L of fuel needed to travel 560 km
- **d)** speed of a falling object: 10 m/s or 60 km/h

Lesson 8.2

3. Based on her best consistent pace in practice, Petra believes that she can run her next marathon at an average pace of 3.75 min/km. An official marathon course is 42.195 km long. To qualify for the Boston Marathon, she must run a marathon in 3 h 40 min or better. If she manages to maintain her target pace throughout her next marathon, will Petra qualify to run in the Boston Marathon?

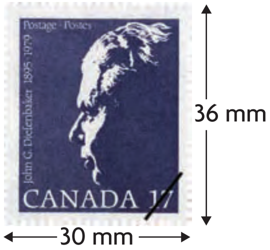
4. Doris wants to buy new carpet for her living room, which has the dimensions shown. She plans to order about 10% extra, so that she has enough to allow for loss during cutting. She can buy the carpet locally for \$36.95 per square yard, or she can buy it from a store in a nearby city for \$33.99 per square yard. However, the store in the nearby city charges \$100 for delivery.



- **a)** How much carpet should Doris buy?
- **b)** Where should she buy the carpet? Explain.

Lesson 8.3

5. In Humboldt, Saskatchewan, there is a 2.4 m by 1.8 m reproduction of a stamp that honours John Diefenbaker, Canada's 13th prime minister. What scale factor was used to make the reproduction?



6. Find a 2-D shape in your classroom, and measure its dimensions.

- **a)** Determine a reasonable scale factor you can use to create a scale diagram on half of a sheet of standard paper.
- **b)** Draw a scale diagram of your shape.

7. An airplane starts at point A. It flies N30°E, at a speed of 125 mph, for 5 h to point B. Then it flies S20°E, at a speed of 100 mph, for 3 h to point C.

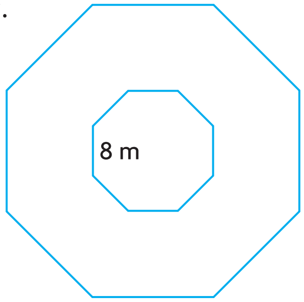
- **a)** Make a scale drawing of the airplane's flight path.
- **b)** Explain how you could estimate the distance from point C to point A without using trigonometry.

Lesson 8.4

8. The owners of a local pizzeria advertise their Gynormous pizza as being 40% bigger than their competitors' pizzas. They do not say, however, what they mean by "bigger."

- **a)** If they mean that the diameter is 40% greater, what is the percent increase in area?
- **b)** If they mean that the area is 40% greater, what is the percent increase in diameter?
- **c)** Which of these two meanings do you think was implied by the owners of the pizzeria? Explain.

9. The area of the larger regular octagon is exactly 2500 m^2.



a) Determine the area of the smaller octagon.

b) Determine the scale factor, to the nearest hundredth, that was used to enlarge the smaller regular octagon.

10. a) Suppose that you put a 5 in. by 7 in. picture in a copy machine and click "enlarge 110%." What will the dimensions of the copy be?

b) By what percent will the area of the picture increase?

Lesson 8.5

11. In a local store, Serena saw a toy pig with a scale ratio of 1 : 16. She estimated that the toy pig was about 10 cm long. She searched online and found a similar toy pig, with a scale ratio of 1 : 64. Estimate the length of the online toy.

12. At the Visitor Centre in 100 Mile House, British Columbia, there is a display of a giant pair of cross-country skis. An average person would use skis that are 200 cm long and poles that are 150 cm long. The giant skis are 12.0 m long.

a) Determine the scale factor that was used to create the display.

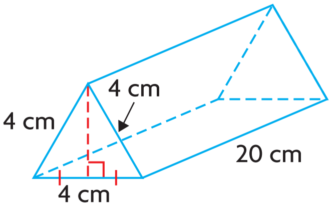
b) Determine the length of the poles in the display.

13. Jonas collects and builds model airplanes. He wants to build a 1 : 20 scale model of a floatplane. He searches the Internet for information about the real plane and learns that it has a wingspan of 36 ft, a length of 26 ft 2 in., and a height of 7 ft 6 in. Jonas wants to build a glass display case, in the form of a rectangular prism, for his scale model. He wants the dimensions of the display case to be 20% larger than each dimension of the model. Determine the dimensions of Jonas's display case.

Lesson 8.6

14. Cone A is a reduction of cone B, with a scale factor of 1 : 9. Cone A has a volume of 20 cm^3. What is the volume of cone B?

15. A chocolate bar is sold in a package, as shown. The manufacturer doubles the volume of the chocolate to create a larger bar, similar in shape to the original bar. Determine the surface area, to the nearest square centimetre, of the package that is needed for the new bar.

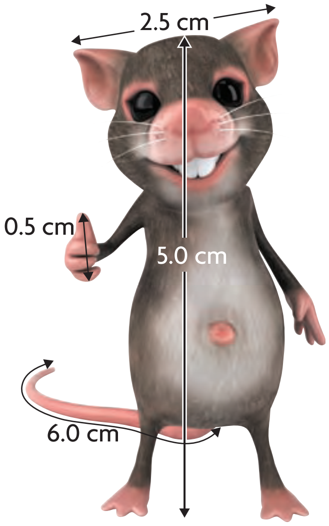


16. A cellphone company advertises that it has created a similar version of its most popular phone, reducing the volume and mass of the original phone by 48.8%. The original phone is a rectangular prism, 50 mm wide by 95 mm long by 10 mm high. Determine the dimensions of the new phone.

8 Chapter Task

Mice To Be Here

A graphic artist is creating computer-generated images for an animated movie. She has to create a school used by mice for the movie. She needs to scale everything that would appear in a regular school down to a size that is appropriate for mice.



❓ What scale factor should she use?

A. Determine the scale factor she should use, based on the relative sizes of the cartoon mice and humans. Explain your thinking.

B. Pick a classroom in your school and some objects that are found in this room. Determine the dimensions of a similar room for the movie and the dimensions of similar objects. Then determine the volume of one similar object that the mice would use in their school, as well as the surface area of another similar object.

C. Based on the dimensions of the similar objects, and assuming that there will be 20 mice in the classroom, do you still believe that your scale factor is appropriate? Explain why or why not. If not, adjust the scale factor and repeat.

D. Normal "travelling" speed for a mouse is 200 cm/s. Determine whether students could walk from one end of a room to the other in the same time that a mouse would travel a similar distance in the room in their school.

E. If the scale factor for the movie were based on the speeds of mice and humans instead of their sizes, what would be the dimensions of the similar room and objects from part B?

Task *Checklist*

- ✔ Did you include your scale factors?
- ✔ Did you show your work for all the calculations?
- ✔ Did you justify and explain your thinking clearly?

8 Project Connection

Peer Critiquing of Research Projects

Now that you have completed your research for your question/topic, prepared your report and presentation, and assessed your own work, you are ready to see and hear the research projects developed by your classmates. You will not be a passive observer, however. You will have an important role—to provide constructive feedback to your peers about their projects and their presentations.

Critiquing a project does *not* involve commenting on what might have been or should have been. It involves reacting to what you see and hear. For example, pay attention to
- strengths and weaknesses of the presentation.
- problems or concerns with the presentation.

Think of suggestions you could make to help your classmate improve future presentations.

While observing each presentation, consider the content, the organization, and the delivery. Take notes during the presentation, using the following rating scales as a guide. You can also use these scales to evaluate the presentation.

Content

Shows a clear sense of audience and purpose.	5	4	3	2	1
Demonstrates a thorough understanding of the topic.	5	4	3	2	1
Clearly and concisely explains ideas.	5	4	3	2	1
Applies mathematical knowledge and skills developed in this course.	5	4	3	2	1
Justifies conclusions with sound mathematical reasoning.	5	4	3	2	1
Uses mathematical terms and symbols correctly.	5	4	3	2	1

Organization

Presentation is clearly focused.	5	4	3	2	1
Engaging introduction includes the research question, clearly stated.	5	4	3	2	1
Key ideas and information are logically presented.	5	4	3	2	1
There are effective transitions between ideas and information.	5	4	3	2	1
Conclusion follows logically from the analysis and relates to the question.	5	4	3	2	1

Delivery

Speaking voice is clear, relaxed, and audible.	5	4	3	2	1
Pacing is appropriate and effective for the allotted time.	5	4	3	2	1
Technology is used effectively.	5	4	3	2	1
Visuals and handouts are easily understood.	5	4	3	2	1
Responses to audience's questions show a thorough understanding of the topic.	5	4	3	2	1

Keep these rating scales in mind as you prepare your own presentation.

Your Turn

A. Think of an excellent presentation you have observed. List five aspects of this presentation that made it effective.

B. What are some common difficulties that students encounter when giving presentations to their peers? Suggest how each of these difficulties could be avoided or dealt with.

C. When critiquing a presentation, it is essential that you do not allow your own opinions about the topic to influence your analysis. Explain why.

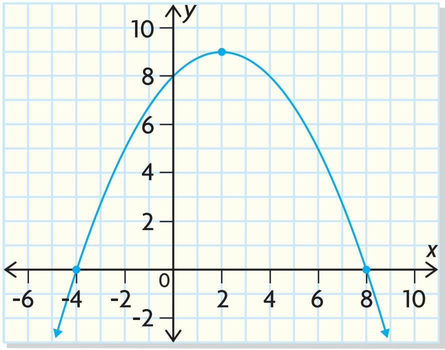
6-8 Cumulative Review

1. i) Express each quadratic function in factored form, and determine
- the x-intercepts.
- the equation of the axis of symmetry.
- the coordinates of the vertex.
- the y-intercept.

ii) Sketch the graph of each function.
iii) State the domain and range of each function.
a) $f(x) = x^2 - 8x$ **b)** $y = x^2 + 2x - 15$

2. i) For each quadratic function, determine
- the equation of the axis of symmetry.
- the coordinates of the vertex.
- the y-intercept.

ii) Sketch the graph of each function.
iii) State the domain and range of each function.
a) $y = -2(x - 3)^2 + 8$ **b)** $g(x) = 0.5\,(x + 2)^2 - 1$



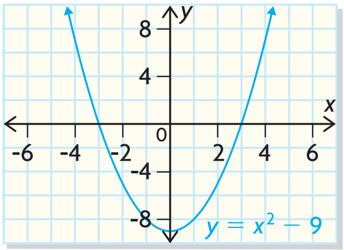
3. Determine the equation of the quadratic function shown to the left. Express the equation in factored form and standard form.

4. The parabola $f(x) = 4x^2 + 24x + 31$ has $x = -3$ as its axis of symmetry.
a) Does the parabola open up or down? Explain.
b) Determine the coordinates of the vertex of the parabola.
c) Does the parabola have a maximum value or a minimum value? Explain how you know.

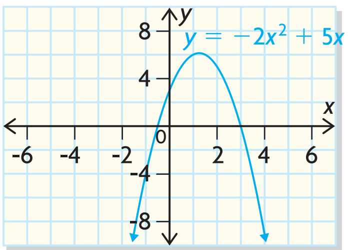
5. Ryan owns a small music store. He currently charges $10 for each CD. At this price, he sells about 80 CDs a week. Experience has taught him that a $1 increase in the price of a CD means a drop of about five CDs per week in sales. At what price should Ryan sell his CDs to maximize his revenue?

6. Determine the roots to the quadratic equation, where $y = 0$.

a)



b)



7. Solve each equation.
a) $(x - 5)(2x + 1) = 0$
b) $x^2 - 4x - 32 = 0$
c) $3x^2 - 10x = 8$
d) $x^2 - 6x - 10 = 0$
e) $2(x - 3)^2 - 8 = 0$
f) $1.5x^2 = 6.1x - 1.1$

8. Two skydivers jump out of an airplane at an altitude of 4.5 km. Their altitude, in metres, is modelled by the function $h(t) = 4500 - 5t^2$, where t is the time in seconds after jumping out of the airplane.

a) Determine the altitude of the skydivers after 5 s.

b) The skydivers opened their parachutes at an altitude of 1500 m. Determine how long they were in free fall.

9. A rapid-transit company currently has 5000 passengers daily, each paying a fare of \$2.25. The company estimates that for each \$0.50 increase in the fare, it will lose 150 passengers daily. The company must have \$15 275 in revenue each day to stay in business. Determine the minimum fare that the company can charge to produce this revenue.

10. State the restrictions on the variable in each equation, and then solve the equation.

a) $9 = \sqrt{x - 1} + 1$ **b)** $-3 = \sqrt{5y + 1}$

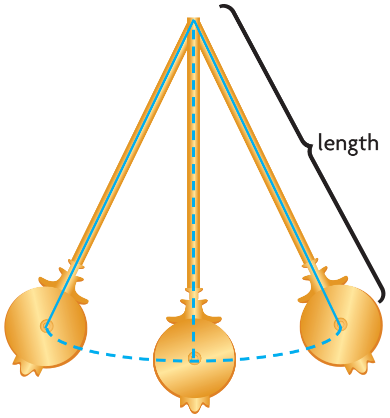
11. The time T, in seconds, for a pendulum to swing back and forth is called its period. The formula that is used to calculate the period of a pendulum is

$$T = 2p\sqrt{\frac{L}{g}},$$

where L is the length of the pendulum in metres

and g is the acceleration due to gravity, 9.81 m/s^2.

The pendulum on a grandfather clock has a period of 2 s. Calculate the length of the pendulum.



12. Carol can read at a rate of one page per minute. Joanna can read at a rate of one page every 40 s. Both girls need to read a 180-page story for English class. How much longer will Carol take to read the story?

13. In an aerial photograph, a rectangular plot of land measures 3 cm by 5 cm. The scale ratio used in the photograph is 1 : 50 000.

a) Determine the dimensions of the plot of land.

b) Determine the area of the plot of land in hectares. (The conversion rate is 1 ha/10 000 m^2.)

14. A new underground holding tank for storm runoff is going to be built as part of a town's waste-water management system. The new tank will be similar to the current tank. The interior dimensions of the current tank are shown here. Engineers estimate that the capacity of the new tank needs to be 10 times the capacity of the current tank. Estimate the interior dimensions of the new tank.